特殊関数要覧

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目次

1 定数

1.1 円周率

Definition 1.1.

$$\pi := 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = 3.14159265358979323846264338327 95\ldots$$

Theorem 1.2 (積分)**.**

$$\pi = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$
$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$
$$\pi = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$$
$$\pi = \int_{-\infty}^{\infty} \frac{1}{\cosh x}\,dx$$

Theorem 1.3 (Machin-like formula)**.**

$$\frac{\pi}{4} = \arctan 1$$
$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$
$$\frac{\pi}{4} = 2\arctan\frac{1}{2} - \arctan\frac{1}{7}$$
$$\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7}$$
$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$
$$\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}$$
$$\frac{\pi}{4} = 6\arctan\frac{1}{8} + 2\arctan\frac{1}{57} + \arctan\frac{1}{239}$$
$$\frac{\pi}{4} = 12\arctan\frac{1}{49} + 32\arctan\frac{1}{57} - 5\arctan\frac{1}{239} + 12\arctan\frac{1}{110443}$$
$$\frac{\pi}{4} = 44\arctan\frac{1}{57} + 7\arctan\frac{1}{239} - 12\arctan\frac{1}{682} + 24\arctan\frac{1}{12943}$$

Theorem 1.4 (連分数)**.**

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \ddots}}}}}$$

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \ddots}}}}$$

Theorem 1.5 (BBP-type formulas)**.**

$$\pi = 4\sum_{0\le n}\frac{(-1)^n}{2n+1}$$

$$\pi = 3\sum_{0\le n}(-1)^n\left(\frac{1}{6n+1}+\frac{1}{6n+5}\right)$$

$$\pi = 4\sum_{0\le n}(-1)^n\left(\frac{1}{10n+1}-\frac{1}{10n+3}+\frac{1}{10n+5}-\frac{1}{10n+7}+\frac{1}{10n+9}\right)$$

$$\pi = \sum_{0\le n}(-1)^n\left(\frac{3}{14n+1}-\frac{3}{14n+3}+\frac{3}{14n+5}+\frac{4}{14n+7}\right.$$
$$\left.+\frac{4}{14n+9}-\frac{4}{14n+11}+\frac{4}{14n+13}\right)$$

$$\pi = \sum_{0\le n}(-1)^n\left(\frac{2}{18n+1}+\frac{3}{18n+3}+\frac{2}{18n+5}-\frac{2}{18n+7}\right.$$
$$\left.-\frac{2}{18n+11}+\frac{2}{18n+13}+\frac{3}{18n+15}+\frac{2}{18n+17}\right)$$

$$\pi = \sum_{0\le n}(-1)^n\left(\frac{3}{22n+1}-\frac{3}{22n+3}+\frac{3}{22n+5}-\frac{3}{22n+7}+\frac{3}{22n+9}\right.$$
$$\left.+\frac{8}{22n+11}+\frac{3}{22n+13}-\frac{3}{22n+15}+\frac{3}{22n+17}-\frac{3}{22n+19}+\frac{1}{22n+21}\right)$$

$$\pi = \sum_{0\le n}\frac{1}{16^n}\left(\frac{4}{8n+1}-\frac{2}{8n+4}-\frac{1}{8n+5}-\frac{1}{8n+6}\right)$$

$$\pi = \frac{1}{2}\sum_{0\le n}\frac{1}{16^n}\left(\frac{8}{8n+2}+\frac{4}{8n+3}+\frac{4}{8n+4}-\frac{1}{8n+7}\right)$$

$$\pi = \frac{1}{16}\sum_{0\le n}\frac{1}{2^{8n}}\left(\frac{2^6}{16n+1}-\frac{2^5}{16n+4}-\frac{2^4}{16n+5}-\frac{2^4}{16n+6}\right.$$
$$\left.+\frac{2^2}{16n+9}-\frac{2}{16n+12}-\frac{1}{16n+13}-\frac{1}{16n+14}\right)$$

$$\pi = \frac{1}{32}\sum_{0\le n}\frac{1}{2^{8n}}\left(\frac{2^7}{16n+2}+\frac{2^6}{16n+3}+\frac{2^6}{16n+4}-\frac{2^4}{16n+7}\right.$$
$$\left.+\frac{2^3}{16n+10}+\frac{2^2}{16n+11}+\frac{2^2}{16n+12}-\frac{1}{16n+15}\right)$$

$$\pi = \frac{1}{32}\sum_{0\le n}\frac{1}{2^{12n}}\left(\frac{2^8}{24n+2}+\frac{3\cdot 2^6}{24n+3}-\frac{2^8}{24n+4}-\frac{3\cdot 2^5}{24n+6}-\frac{3\cdot 2^5}{24n+8}\right.$$
$$\left.+\frac{2^4}{24n+10}-\frac{2^2}{24n+12}-\frac{3}{24n+15}-\frac{3\cdot 2}{24n+16}-\frac{2}{24n+18}-\frac{1}{24n+20}\right)$$

$$\pi = \frac{1}{64}\sum_{0\le n}\frac{1}{2^{12n}}\left(\frac{2^8}{24n+1}+\frac{2^8}{24n+2}-\frac{3\cdot 2^7}{24n+3}-\frac{2^8}{24n+4}\right.$$
$$-\frac{2^6}{24n+5}+\frac{3\cdot 2^5}{24n+8}+\frac{2^6}{24n+9}+\frac{2^4}{24n+10}+\frac{2^3}{24n+12}-\frac{2^2}{24n+13}$$
$$\left.+\frac{3\cdot 2}{24n+15}+\frac{3\cdot 2}{24n+16}+\frac{1}{24n+17}+\frac{1}{24n+18}-\frac{1}{24n+20}-\frac{1}{24n+21}\right)$$

$$\pi = \frac{1}{3\cdot 2^5}\sum_{0\le n}\frac{1}{2^{12n}}\left(\frac{2^8}{24n+2}+\frac{2^6}{24n+3}+\frac{2^7}{24n+5}+\frac{11\cdot 2^5}{24n+6}\right.$$
$$+\frac{2^6}{24n+7}+\frac{9\cdot 2^5}{24n+8}+\frac{2^7}{24n+9}+\frac{5\cdot 2^4}{24n+10}+\frac{5\cdot 2^2}{24n+12}-\frac{2^4}{24n+14}$$
$$\left.-\frac{1}{24n+15}+\frac{3\cdot 2}{24n+16}-\frac{2}{24n+17}-\frac{1}{24n+19}+\frac{1}{24n+20}-\frac{2}{24n+21}\right)$$

$$\pi = \frac{1}{3\cdot 2^5}\sum_{0\le n}\frac{1}{2^{12n}}\left(\frac{2^8}{24n+1}+\frac{5\cdot 2^6}{24n+3}+\frac{2^8}{24n+4}-\frac{3\cdot 2^6}{24n+5}-\frac{7\cdot 2^5}{24n+6}\right.$$
$$-\frac{2^6}{24n+7}-\frac{3\cdot 2^6}{24n+8}-\frac{2^6}{24n+9}-\frac{2^6}{24n+10}-\frac{7\cdot 2^2}{24+12}-\frac{2^2}{24n+13}$$
$$\left.-\frac{5}{24n+15}+\frac{3}{24n+17}+\frac{1}{24n+18}+\frac{1}{24n+19}+\frac{1}{24n+21}-\frac{1}{24n+22}\right)$$

$$\pi = \frac{1}{3\cdot 2^5}\sum_{0\le n}\frac{1}{2^{12n}}\left(\frac{2^9}{24n+1}-\frac{2^8}{24n+2}+\frac{2^6}{24n+3}-\frac{2^9}{24n+4}-\frac{2^5}{24n+6}+\frac{2^6}{24n+7}\right.$$
$$+\frac{3\cdot 2^5}{24n+8}+\frac{2^6}{24n+9}+\frac{3\cdot 2^4}{24n+10}-\frac{3\cdot 2^2}{24n+12}-\frac{2^3}{24n+13}-\frac{2^4}{24n+14}$$
$$\left.-\frac{1}{24n+15}-\frac{3\cdot 2}{24n+16}-\frac{2}{24n+18}-\frac{1}{24n+19}-\frac{1}{24n+20}-\frac{1}{24n+21}\right)$$

$$
\begin{aligned}
\pi = \frac{1}{2^{12}} \sum_{0\le n} \frac{1}{2^{16n}} \Bigg(& \frac{2^{14}}{32n+1} - \frac{2^{13}}{32n+4} - \frac{2^{12}}{32n+5} - \frac{2^{12}}{32n+6} \\
& + \frac{2^{10}}{32n+9} - \frac{2^{9}}{32n+12} - \frac{2^{8}}{32n+13} - \frac{2^{8}}{32n+14} + \frac{2^{6}}{32n+17} - \frac{2^{5}}{32n+20} \\
& - \frac{2^{4}}{32n+21} - \frac{2^{4}}{32n+22} + \frac{2^{2}}{32n+25} - \frac{2}{32n+28} - \frac{1}{32n+29} - \frac{1}{32n+30} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\pi = \frac{1}{2^{12}} \sum_{0\le n} \frac{1}{2^{16n}} \Bigg(& \frac{2^{15}}{32n+2} + \frac{2^{14}}{32n+3} + \frac{2^{14}}{32n+4} - \frac{2^{12}}{32n+7} \\
& + \frac{2^{11}}{32n+10} + \frac{2^{10}}{32n+11} + \frac{2^{10}}{3n+12} - \frac{2^{8}}{32n+15} + \frac{2^{7}}{32n+18} + \frac{2^{6}}{32n+19} \\
& + \frac{2^{6}}{32n+20} - \frac{2^{4}}{32n+23} + \frac{2^{3}}{32n+26} + \frac{2^{2}}{32n+27} + \frac{2^{2}}{32n+28} - \frac{1}{32n+31} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\pi = \frac{1}{2^{6}} \sum_{0\le n} \frac{(-1)^n}{2^{10n}} \Bigg(& -\frac{2^{5}}{4n+1} - \frac{1}{4n+3} + \frac{2^{8}}{10n+1} - \frac{2^{6}}{10n+3} \\
& - \frac{2^{2}}{10n+5} - \frac{2^{2}}{10n+7} + \frac{1}{10n+9} \Bigg)
\end{aligned}
$$

$$
\pi = \frac{1}{\sqrt{2}} \sum_{0\le n} \frac{(-1)^n}{8^n} \left(\frac{4}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} \right)
$$

$$
\begin{aligned}
\pi = \frac{1}{2^{6}\sqrt{2}} \sum_{0\le n} \frac{(-1)^n}{2^{9n}} \Bigg(& \frac{2^{8}}{18n+1} + \frac{2^{6}}{18n+3} + \frac{2^{6}}{18n+5} - \frac{2^{5}}{18n+7} \\
& - \frac{2^{3}}{18n+9} - \frac{2^{3}}{18n+11} + \frac{2^{2}}{18n+13} + \frac{1}{18n+15} + \frac{1}{18n+17} \Bigg)
\end{aligned}
$$

$$
\pi = 2\sqrt{2} \sum_{0\le n} (-1)^n \left(\frac{1}{4n+1} + \frac{1}{4n+3} \right)
$$

$$
\begin{aligned}
\pi = 2\sqrt{2} \sum_{0\le n} (-1)^n \Bigg(& \frac{1}{12n+1} + \frac{1}{12n+3} - \frac{1}{12n+5} - \frac{1}{12n+7} \\
& + \frac{1}{12n+9} + \frac{1}{12n+11} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\pi = \frac{1}{\sqrt{2}} \sum_{0\le n} (-1)^n \Bigg(& \frac{3}{20n+1} + \frac{3}{20n+3} + \frac{2}{20n+5} - \frac{3}{20n+7} \\
& + \frac{3}{20n+9} + \frac{3}{20n+11} - \frac{3}{20n+13} + \frac{2}{20n+17} + \frac{3}{20n+19} \Bigg)
\end{aligned}
$$

$$\pi = \frac{1}{8\sqrt{2}} \sum_{0 \le n} \frac{1}{2^{6n}} \left(\frac{2^5}{12n+1} + \frac{2^3}{12n+3} + \frac{2^3}{12n+5} - \frac{2^2}{12n+7} \right.$$
$$\left. - \frac{1}{12n+9} - \frac{1}{12n+11} \right)$$
$$\pi = \frac{1}{4\sqrt{3}} \sum_{0 \le n} \frac{1}{64^n} \left(\frac{20}{6n+1} + \frac{6}{6n+2} - \frac{1}{6n+3} - \frac{3}{6n+4} - \frac{1}{6n+5} \right)$$
$$\pi = \frac{1}{9\sqrt{3}} \sum_{0 \le n} \frac{1}{3^{6n}} \left(\frac{3^4}{12n+1} - \frac{2 \cdot 3^3}{12n+2} - \frac{3^2}{12n+4} - \frac{4 \cdot 3}{12n+6} \right.$$
$$\left. - \frac{3}{12n+7} - \frac{2}{12n+8} - \frac{1}{12n+10} \right)$$
$$\pi = \frac{1}{36\sqrt{3}} \sum_{0 \le n} \frac{1}{3^{6n}} \left(\frac{3^4}{12n+1} + \frac{3^3}{12n+2} - \frac{2 \cdot 3^4}{12n+3} - \frac{3^2}{12n+4} + \frac{3^3}{12n+5} \right.$$
$$\left. + \frac{8 \cdot 3}{12n+6} - \frac{3}{12n+7} + \frac{7}{12n+8} + \frac{2 \cdot 3}{12n+9} + \frac{3}{12n+10} - \frac{1}{12n+11} \right)$$
$$\pi = \frac{1}{9\sqrt{3}} \sum_{0 \le n} \frac{1}{3^{6n}} \left(\frac{3^4}{12n+1} + \frac{7 \cdot 3^3}{12n+2} + \frac{5 \cdot 3^2}{12n+4} + \frac{3^3}{12n+5} \right.$$
$$\left. + \frac{8 \cdot 3}{12n+6} - \frac{3}{12n+7} + \frac{1}{12n+8} + \frac{1}{12n+10} - \frac{1}{12n+11} \right)$$
$$\pi = 50 \sum_{0 \le n} \frac{1}{\phi^{5n}} \left(\frac{\phi^{-2}}{(5n+1)^2} - \frac{\phi^{-1}}{(5n+2)^2} - \frac{\pi^{-2}}{(5n+3)^2} + \frac{\phi^{-5}}{(5n+4)^2} + \frac{2\phi^{-5}}{(5n+5)^2} \right)$$
$$\pi^2 = \frac{9}{8} \sum_{0 \le n} \frac{1}{64^n} \left(\frac{16}{(6n+1)^2} - \frac{24}{(6n+2)^2} - \frac{8}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{1}{(6n+5)^2} \right)$$
$$\pi^2 = \frac{2}{27} \sum_{0 \le n} \frac{1}{3^{6n}} \left(\frac{3^5}{(12n+1)^2} - \frac{5 \cdot 3^4}{(12n+2)^2} - \frac{3^4}{(12n+4)^2} - \frac{3^3}{(12n+5)^2} \right.$$
$$\left. - \frac{8 \cdot 3^2}{(12n+6)^2} - \frac{3^2}{(12n+7)^2} - \frac{3^2}{(12n+8)^2} - \frac{5}{(12n+10)^2} + \frac{1}{(12n+11)^2} \right)$$

Theorem 1.6 (Ramanujan-Sato series)**.**

$$\frac{1}{\pi} = \sum_{0 \le n} \frac{(2n)!^3}{n!^6} \frac{6n+1}{2^{8n+2}}$$

$$\frac{1}{\pi} = \sum_{0 \le n} \frac{(2n)!^3}{n!^6} \frac{42n+5}{2^{12n+4}}$$

$$\frac{1}{\pi} = \sum_{0 \le n} \frac{(2n)!^3}{n!^6} \frac{5\sqrt{5}-1+(30+42\sqrt{5})n}{2^{20n+5}} (\sqrt{5}-1)^{8n}$$

$$\frac{1}{\pi} = \sum_{0 \le n} \frac{(2n)!(3n)!}{n!^3} \frac{15n+2}{2^{n-2}3^{6n+3}}$$

$$\frac{1}{\pi} = \frac{2}{\sqrt{3}} \sum_{0 \le n} \frac{(2n)!(3n)!}{n!^5} \frac{33n+4}{15^{3n+1}}$$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2} \sum_{0 \le n} \frac{(4n)!}{n!^4} \frac{1103+26390n}{396^{4n}}$$

$$\frac{1}{\pi} = 12 \sum_{0 \le n} \frac{(-1)^n (6n)!}{(3n)! n!^3} \frac{13591409+545140134n}{(640320)^{3n+3/2}}$$

1.2 Napier 数

Definition 1.7.

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828182845904523536028747135 2\ldots$$

Theorem 1.8.

$$e = \sum_{0 \le n} \frac{1}{n!}$$

$$e = \frac{1}{2} \sum_{0 < n} \frac{n}{(n-1)!}$$

$$e = 2 \sum_{0 < n} \frac{n}{(2n-1)!}$$

$$e = \sum_{0 \le n} \frac{(3n)^2 + 1}{(3n)!}$$

$$e = \frac{1}{2} \sum_{0 < n} \frac{n^2}{n!}$$

$$e = \frac{1}{5} \sum_{0 < n} \frac{n^3}{n!}$$

Theorem 1.9 (連分数)**.**

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}}$$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \ddots}}}}}$$

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cfrac{6}{6 + \ddots}}}}}$$

$$e = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \ddots}}}}}$$

Theorem 1.10 (Pippenger type product)**.**

1.3 Euler の定数

Definition 1.11.

$$\gamma := \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0.577215664901532860606512090\ldots$$

2　初等超越関数

2.1　指数関数

Definition 2.1.

$$e^x := \sum_{0 \le n} \frac{x^n}{n!}$$

Theorem 2.2 (特殊値)**.**

$$\begin{aligned}
e^{2\pi i} &= 1 \\
e^{\pi i} &= -1 \\
e^{\pi i/2} &= i \\
e^{\pi i/3} &= \frac{1}{2} + i\frac{\sqrt{3}}{2} \\
e^{2\pi i/3} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\
e^{ix} &= \cos x + i \sin x
\end{aligned}$$

Theorem 2.3.

$$e^{x+y} = e^x \cdot e^y$$

Theorem 2.4.

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

2.2　双曲線関数

Definition 2.5.

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x}
\end{aligned}$$

Theorem 2.6.

$$\cosh x + \sinh x = e^x$$
$$\cosh x - \sinh x = e^{-x}$$
$$\cosh^2 x - \sinh^2 x = 1$$

Theorem 2.7.

$$\sinh ix = i \sin x$$
$$\cosh ix = \cos x$$
$$\tanh ix = i \tan x$$

Theorem 2.8 (微分)**.**

$$\frac{d}{dx} \sinh x = \cosh x$$
$$\frac{d}{dx} \cosh x = \sinh x$$
$$\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x}$$
$$\frac{d}{dx} \coth x = -\frac{1}{\sinh^2 x}$$

Theorem 2.9 (級数表示)**.**

$$\sinh x = \sum_{0 \le n} \frac{1}{(2n+1)!} x^{2n+1}$$
$$\cosh x = \sum_{0 \le n} \frac{1}{(2n)!} x^{2n}$$
$$\tanh x = \sum_{0 < n} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1}$$
$$\coth x = \sum_{0 \le n} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1}$$
$$\frac{1}{\sinh x} = 2 \sum_{0 \le n} \frac{(1-2^{2n-1})B_{2n}}{(2n)!} x^{2n-1}$$
$$\frac{1}{\cosh x} = \sum_{0 \le n} \frac{E_{2n}}{(2n)!} x^{2n}$$

Theorem 2.10 (無限乗積展開)**.**

$$\sinh \pi x = \pi x \prod_{0<n} \left(1 + \frac{x^2}{n^2}\right)$$
$$\cosh \pi x = \prod_{0\leq n} \left(1 + \frac{x^2}{\left(n+\frac{1}{2}\right)^2}\right)$$

Theorem 2.11 (部分分数展開)**.**

$$\pi \coth \pi x = x \sum_{n=-\infty}^{\infty} \frac{1}{n^2+x^2}$$
$$\pi \tanh \pi x = x \sum_{n=-\infty}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2+x^2}$$
$$\frac{\pi}{\sinh \pi x} = x \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+x^2}$$
$$\frac{\pi}{\cosh \pi x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)^2+x^2}$$

Theorem 2.12 (加法定理)**.**

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x + \sinh y$$
$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

Theorem 2.13 (倍角公式)**.**

$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = 2\cosh^2 x - 1$$
$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$

2.3 三角関数

Definition 2.14.

$$\sin x := \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos x := \frac{e^{ix} + e^{-ix}}{2}$$
$$\tan x := \frac{\sin x}{\cos x}$$
$$\cot x := \frac{\cos x}{\sin x}$$

Theorem 2.15 ($\sin x$ の特殊値)**.**

$$\sin 0 = 0$$
$$\sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$$
$$\sin \frac{\pi}{10} = \frac{\sqrt{5} - 1}{4}$$
$$\sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$$
$$\sin \frac{\pi}{6} = \frac{1}{2}$$
$$\sin \frac{\pi}{5} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$$
$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\sin \frac{3\pi}{10} = \frac{1 + \sqrt{5}}{4}$$
$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\sin \frac{3\pi}{8} = \frac{2 + \sqrt{2}}{2}$$
$$\sin \frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$
$$\sin \frac{5\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{2}$$
$$\sin \frac{\pi}{2} = 1$$

Theorem 2.16 ($\tan x$ の特殊値)**.**

$$
\begin{aligned}
&\tan 0 = 0 \\
&\tan \frac{\pi}{12} = 2 - \sqrt{3} \\
&\tan \frac{\pi}{10} = \sqrt{1 - \frac{2}{\sqrt{5}}} \\
&\tan \frac{\pi}{8} = \sqrt{2} - 1 \\
&\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \\
&\tan \frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}} \\
&\tan \frac{\pi}{4} = 1 \\
&\tan \frac{3\pi}{10} = \sqrt{1 + \frac{2}{\sqrt{5}}} \\
&\tan \frac{\pi}{3} = \sqrt{3} \\
&\tan \frac{3\pi}{8} = \sqrt{2} + 1 \\
&\tan \frac{2\pi}{5} = \sqrt{5 + 2\sqrt{5}} \\
&\tan \frac{5\pi}{12} = 2 + \sqrt{3} \\
&\tan \frac{\pi}{2} = \infty
\end{aligned}
$$

Theorem 2.17.

$$\cos^2 x + \sin^2 x = 1$$

Theorem 2.18 (微分)**.**

$$
\begin{aligned}
&\frac{d}{dx} \sin x = \cos x \\
&\frac{d}{dx} \cos x = -\sin x \\
&\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} \\
&\frac{d}{dx} \cot x = -\frac{1}{\sin^2 x}
\end{aligned}
$$

Theorem 2.19.

$$\int \sin x \, dx = -\cos x + C$$
$$\int \cos x \, dx = \sin x + C$$
$$\int \tan x \, dx = -\ln|\cos x| + C$$
$$\int \cot x \, dx = \ln|\sin x| + C$$
$$\int \frac{1}{\sin x} \, dx = \frac{1}{2} \ln\left(\frac{1-\cos x}{1+\cos x}\right) + C$$
$$\int \frac{1}{\cos x} \, dx = \frac{1}{2} \ln\left(\frac{1+\sin x}{1-\sin x}\right) + C$$

Theorem 2.20 (級数表示)**.**

$$\sin x = \sum_{0 \le n} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$\cos x = \sum_{0 \le n} \frac{(-1)^n}{(2n)!} x^{2n}$$
$$\tan x = \sum_{0 < n} \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n}}{(2n)!} x^{2n-1}$$
$$\cot x = \sum_{0 \le n} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1}$$
$$\frac{1}{\sin x} = 2 \sum_{0 \le n} \frac{(-1)^{n-1} (2^{2n-1}-1) B_{2n}}{(2n)!} x^{2n-1}$$
$$\frac{1}{\cos x} = \sum_{0 \le n} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}$$

Theorem 2.21 (無限乗積展開)**.**

$$\sin \pi x = \pi x \prod_{0 < n} \left(1 - \frac{x^2}{n^2}\right)$$
$$\cos \pi x = \prod_{0 \le n} \left(1 - \frac{x^2}{\left(n + \frac{1}{2}\right)^2}\right)$$

Theorem 2.22 (部分分数展開)**.**

$$\pi \cot \pi x = x \sum_{n=-\infty}^{\infty} \frac{1}{x^2 - n^2}$$

$$\pi \tan \pi x = x \sum_{n=-\infty}^{\infty} \frac{1}{\left(n + \frac{1}{2}\right)^2 - x^2}$$

$$\frac{\pi}{\sin \pi x} = x \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x^2 - n^2}$$

$$\frac{\pi}{\cos \pi x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \left(n + \frac{1}{2}\right)}{\left(n + \frac{1}{2}\right)^2 - x^2}$$

Theorem 2.23 (加法定理)**.**

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan \frac{x+y}{2} = \frac{\sin x + \sin y}{\cos x + \cos y}$$

Theorem 2.24 (倍角公式)**.**

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = 2 \cos^2 x - 1$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\cos 3x = 4 \cos^3 x - 3 \cos x$$

$$\sin nx = \sum_{k=0}^{n} \binom{n}{k} \sin^k x \cos^{n-k} x \sin \frac{\pi k}{2}$$

$$\cos nx = \sum_{k=0}^{n} \binom{n}{k} \sin^k x \cos^{n-k} x \cos \frac{\pi k}{2}$$

$$\tan nx = \frac{\sum_{k=0}^{n} \binom{n}{k} \tan^k x \sin \frac{\pi k}{2}}{\sum_{k=0}^{n} \binom{n}{k} \tan^k x \cos \frac{\pi k}{2}}$$

Theorem 2.25 (和積公式).

$$\begin{aligned}\sin x + \sin y &= 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\\ \sin x - \sin y &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}\\ \cos x + \cos y &= 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\\ \cos x - \cos y &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}\end{aligned}$$

Theorem 2.26 (積和公式).

$$\begin{aligned}\sin x\sin y &= -\frac{1}{2}(\cos(x+y)-\cos(x-y))\\ \sin x\cos y &= \frac{1}{2}(\sin(x+y)+\sin(x-y))\\ \cos x\cos y &= \frac{1}{2}(\cos(x+y)+\cos(x-y))\end{aligned}$$

Theorem 2.27 (乗法定理).

$$2^{n-1}\prod_{k=0}^{n-1}\sin\left(x+\frac{\pi k}{n}\right) = \sin nx$$

2.4 対数関数

Definition 2.28.

$$\ln x := \int_0^x \frac{1}{t}\,dt$$

Theorem 2.29 (特殊値).

$$\begin{aligned}\ln 1 &= 0\\ \ln e &= 1\end{aligned}$$

Theorem 2.30 (微分).

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Theorem 2.31.

$$\ln xy = \ln x + \ln y$$

Theorem 2.32 (Taylor 級数).

$$\ln(1+x) = \sum_{0<n}\frac{(-1)^{n-1}}{n}x^n$$

2.5 逆双曲線関数

Definition 2.33.

$$\operatorname{arsinh} x := \int_0^x \frac{1}{\sqrt{1+t^2}}\, dt$$
$$\operatorname{arcosh} x := \int_1^x \frac{1}{\sqrt{t^2-1}}\, dt$$
$$\operatorname{artanh} x := \int_0^x \frac{1}{1-t^2}\, dt$$

Theorem 2.34 (微分)**.**

$$\frac{d}{dx}\operatorname{arsinh} x = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}\operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}\operatorname{artanh} x = \frac{1}{1-x^2}$$

Theorem 2.35.

$$\operatorname{arsinh} x = \ln(x+\sqrt{1+x^2})$$
$$\operatorname{arcosh} x = \ln(x+\sqrt{x^2-1})$$
$$\operatorname{artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

Theorem 2.36 (級数表示)**.**

$$\operatorname{arsinh} x = \sum_{0\le n}\binom{2n}{n}\frac{(-1)^n x^{2n+1}}{2^{2n}(2n+1)}$$
$$\operatorname{artanh} x = \sum_{0\le n}\frac{x^{2n+1}}{2n+1}$$
$$\operatorname{arsinh}^2 x = \frac{1}{2}\sum_{0<n}\frac{(-2x)^{2n}}{n^2\binom{2n}{n}}$$

2.6 逆三角関数

Definition 2.37.

$$
\begin{aligned}
\arcsin x &:= \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt \\
\arccos x &:= \int_x^1 \frac{1}{\sqrt{1-t^2}}\,dt \\
\arctan x &:= \int_0^x \frac{1}{1+t^2}\,dt
\end{aligned}
$$

Theorem 2.38 (微分)**.**

$$
\begin{aligned}
\frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\arccos x &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\arctan x &= \frac{1}{1+x^2}
\end{aligned}
$$

Theorem 2.39.

$$
\begin{aligned}
\arcsin x &= -i\ln(ix+\sqrt{1-x^2}) \\
\arccos x &= -i\ln(x+i\sqrt{1-x^2}) \\
\arctan x &= \frac{i}{2}\ln\frac{1-iz}{1+iz}
\end{aligned}
$$

Theorem 2.40 (級数表示)**.**

$$
\begin{aligned}
\arcsin x &= \sum_{0\le n}\binom{2n}{n}\frac{x^{2n+1}}{2^{2n}(2n+1)} \\
\arctan x &= \sum_{0\le n}\frac{(-1)^n x^{2n+1}}{2n+1} \\
\arcsin^2 x &= \frac{1}{2}\sum_{0<n}\frac{(2x)^{2n}}{n^2\binom{2n}{n}}
\end{aligned}
$$

Theorem 2.41.

$$\sin(a \arcsin x) = a \sum_{0 \le n} \frac{x^{2n+1}}{(2n+1)!} \prod_{k=0}^{n-1} ((2k+1)^2 - a^2)$$

$$\cos(a \arcsin x) = \sum_{0 \le n} \frac{x^{2n}}{(2n)!} \prod_{k=0}^{n-1} ((2k)^2 - a^2)$$

3　ガンマ関数系

3.1　ガンマ関数

Definition 3.1.

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt, \quad (\Re x > 0)$$

一般の複素数に対しては,

$$\Gamma(x) := \lim_{n \to \infty} \frac{n^x n!}{\prod_{k=0}^{n} (x+k)}$$

と定義する.

Theorem 3.2 (特殊値).

$$\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}$$
$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$
$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$
$$\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$$
$$\Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$$
$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$
$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$$
$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$$

Theorem 3.3 (Taylor 級数).

$$\ln\Gamma(1+x) = -\gamma x + \sum_{1<n} \frac{\zeta(n)}{n}(-x)^n$$

Theorem 3.4 (Fourier 級数).

$$\ln\Gamma(x) = \left(\frac{1}{2}-x\right)(\ln 2\pi + \gamma) + \frac{1}{2}\ln\frac{\pi}{\sin \pi x} + \frac{1}{\pi}\sum_{0<n}\frac{\ln n \cdot \sin 2\pi n x}{n}$$

Theorem 3.5 (Weierstrass の乗積表示).

$$\frac{1}{\Gamma(x)} = x e^{\gamma x}\prod_{0<n}\left(1+\frac{x}{n}\right)e^{-x/n}$$

Theorem 3.6 ($x=\infty$ における漸近展開).

$$\ln\Gamma(x) \sim \frac{1}{2}\ln 2\pi + \left(x-\frac{1}{2}\right)\ln x - x + \sum_{0<n}\frac{B_{2n}}{2n(2n-1)x^{2n-1}}$$

Theorem 3.7 (相反公式).

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

Theorem 3.8 (Legendre の倍角公式)**.**

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=2^{1-2x}\sqrt{\pi}\Gamma(2x)$$

Theorem 3.9 (乗法定理)**.**

$$\prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right)=(2\pi)^{(n-1)/2}n^{1/2-nx}\Gamma(nx)$$

Theorem 3.10 (Stirling の公式)**.**

$$\lim_{x\to\infty}\frac{\Gamma(x)}{\sqrt{2\pi}x^{x-1/2}e^{-x}}=1$$

Theorem 3.11.

$$\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}b_k$$

ならば,

$$\lim_{x\to\infty}\frac{\Gamma(x+a_k)}{\Gamma(x+b_k)}=1$$

Theorem 3.12 (Binet's log gamma formulas)**.**

$$\ln\Gamma(x)=\frac{1}{2}+\ln 2\pi+\left(x-\frac{1}{2}\right)\ln x-x+\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-xt}}{t}\,dt$$
$$\ln\Gamma(x)=\frac{1}{2}+\ln 2\pi+\left(x-\frac{1}{2}\right)\ln x-x+\int_0^\infty\frac{\arctan\frac{t}{x}}{e^{2\pi t}-1}\,dt$$

Theorem 3.13 (Raabe 積分)**.**

$$\int_0^x\ln\Gamma(t)\,dt=\frac{x}{2}\ln 2\pi+\frac{x(1-x)}{2}+\ln K(x)$$
$$\int_0^x\ln\Gamma(t)\,dt=\frac{x}{2}\ln 2\pi+\frac{x(1-x)}{2}+x\ln\Gamma(x)-\ln G(1+x)$$

3.2 ベータ関数

Definition 3.14.

$$B(a,b):=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Theorem 3.15 (級数表示).

$$B(a,b) = \sum_{0\le n} \frac{(1-a)_n}{n!(n+b)}$$

Theorem 3.16 (積分表示).

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$
$$B(a,b) = 2\int_0^{\pi/2} \sin^{2a-1} x \cos^{2b-1} x\,dx$$

3.3 ディガンマ関数

Definition 3.17.

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$$

Theorem 3.18 (特殊値).

$$\psi(1) = -\gamma$$
$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2$$
$$\psi\left(\frac{1}{3}\right) = -\gamma - \frac{\pi}{2\sqrt{3}} - \frac{3\ln 3}{2}$$
$$\psi\left(\frac{2}{3}\right) = -\gamma + \frac{\pi}{2\sqrt{3}} - \frac{3\ln 3}{2}$$
$$\psi\left(\frac{1}{4}\right) = -\gamma - \frac{\pi}{2} - 3\ln 2$$
$$\psi\left(\frac{3}{4}\right) = -\gamma + \frac{\pi}{2} - 3\ln 2$$
$$\psi\left(\frac{1}{6}\right) = -\gamma - \frac{\sqrt{3}\pi}{2} - 2\ln 2 - \frac{3\ln 3}{2}$$
$$\psi\left(\frac{1}{8}\right) = -\gamma - \frac{\pi}{2} - 4\ln 2 - \frac{\pi + \ln(2+\sqrt{2}) - \ln(2-\sqrt{2})}{\sqrt{2}}$$
$$\psi(1+x) = -\gamma + H_x$$

Theorem 3.19 (級数表示).

$$\psi(1+x) = -\gamma + \sum_{0<n} \left(\frac{1}{n} - \frac{1}{n+x}\right)$$

Theorem 3.20 (Taylor 級数)**.**

$$\psi(1+x) = -\gamma + \sum_{1<n} (-1)^n \zeta(n) x^{n-1}$$

Theorem 3.21 (相反公式)**.**

$$\psi(1-x) - \psi(x) = \pi \cot \pi x$$

Theorem 3.22 (乗法定理)**.**

$$\frac{1}{n} \sum_{k=0}^{n-1} \psi\left(x + \frac{k}{n}\right) = \psi(nx) - \ln n$$

Theorem 3.23 (Gauss's digamma theorem)**.** 正整数 $0 < a < b$ に対し,

$$\psi\left(\frac{a}{b}\right) = -\gamma - \ln(2b) - \frac{\pi}{2} \cot \frac{\pi a}{b} + 2 \sum_{k=1}^{\lfloor (b-1)/2 \rfloor} \cos \frac{2\pi a k}{b} \ln \sin \frac{\pi a}{b}$$

Theorem 3.24 ($x = \infty$ における漸近展開)**.**

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{0<n} \frac{B_{2n}}{2n x^{2n}}$$

3.4 ポリガンマ関数

Definition 3.25.

$$\psi^{(n)}(x) := \frac{d^n}{dx^n} \psi(x)$$

以下, n を正整数とする.

Theorem 3.26.

$$\psi^{(n)}(1+x) = \psi^{(n)}(x) + \frac{(-1)^n n!}{x^{n+1}}$$

Theorem 3.27.

$$\psi^{(n)}(x) = (-1)^{n-1} n! \zeta(n+1, x)$$

Theorem 3.28 (級数表示)**.**

$$\psi^{(r)}(x) = (-1)^{r-1} r! \sum_{0 \leq n} \frac{1}{(n+x)^{r+1}}$$

Theorem 3.29 (Taylor 級数)**.**

$$\psi^{(r)}(x) = r! \sum_{0<n} (-1)^{n-r} \binom{n+r-1}{r} \zeta(n+r) x^{n-1}$$

Theorem 3.30 (積分表示)**.**

$$\psi^{(n)}(x) = (-1)^{n-1} \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}\, dt$$

Theorem 3.31 ($x = \infty$ における漸近展開)**.**

$$\psi^{(r)}(x) \sim (-1)^{r-1} \left(\frac{(r-1)!}{x^r} + \frac{r!}{2x^{r+1}} + \sum_{0<n} \frac{(2n+r-1)! B_{2n}}{(2n)! z^{2n+r}} \right)$$

Theorem 3.32 (乗法定理)**.**

$$\sum_{k=0}^{n-1} \psi^{(r)} \left(x + \frac{k}{n} \right) = n^{r+1} \psi^{(r)}(nx)$$

3.5 Barnes G 関数

Definition 3.33.

$$G(1+x) := (2\pi)^{x/2} \exp\left(-\frac{x + x^2(1+\gamma)}{2} \right) \prod_{0<n} \left(1 + \frac{x}{n} \right)^n \exp\left(-x + \frac{x^2}{2n} \right)$$

Theorem 3.34 (特殊値)**.**

$$G\left(\frac{1}{2}\right) = \frac{e^{1/8} 2^{1/24}}{A^{3/2} \pi^{1/4}}$$
$$G\left(\frac{3}{2}\right) = \frac{\pi^{1/4} e^{1/8} 2^{1/24}}{A^{3/2}}$$
$$G\left(\frac{5}{2}\right) = \frac{\pi^{3/4} e^{1/8} 2^{1/24}}{2A^{3/2}}$$
$$G\left(\frac{1}{4}\right) = \frac{e^{3/32 - \beta(2)/(4\pi)}}{A^{9/8} \Gamma\left(\frac{1}{4}\right)^{3/4}}$$
$$G\left(\frac{3}{4}\right) = \frac{e^{3/32 + \beta(2)/(4\pi)}}{A^{9/8} \Gamma\left(\frac{1}{4}\right)^{1/4}}$$

Theorem 3.35.

$$G(1+x)=\Gamma(x)G(x)$$

Theorem 3.36.

$$\ln\frac{G(1-x)}{G(1+x)}=-x\ln 2\pi+\pi\int_0^x t\cot\pi t\,dt$$
$$\ln\frac{G(1-x)}{G(1+x)}=x\ln\frac{\sin\pi x}{\pi}+\frac{\mathrm{Cl}_2(2\pi x)}{2\pi}$$

Theorem 3.37 (Taylor 展開)**.**

$$\ln G(1+x)=\frac{x}{2}\ln 2\pi-\frac{x+(1+\gamma)x^2}{2}+\sum_{1<n}\frac{(-1)^n\zeta(n)}{n+1}x^{n+1}$$

Theorem 3.38 ($x=\infty$ における漸近展開)**.**

$$\ln G(1+x)\sim\left(\frac{1}{2}\ln x-\frac{3}{4}\right)x^2+\frac{x}{2}\ln 2\pi-\frac{1}{12}\ln x+\zeta'(-1)+\sum_{0<n}\frac{B_{2n+2}}{4n(n+1)x^{2n}}$$

3.6 Kinkelin K 関数

Definition 3.39.

$$K(x):=\exp(\zeta'(-1,x)-\zeta'(-1))$$

Theorem 3.40 (特殊値)**.**

$$K(1)=1$$
$$K(2)=1$$
$$K(3)=4$$
$$K(4)=108$$
$$K(5)=27648$$
$$K(6)=86400000$$
$$K(n+1)=1^1 2^2 3^3\cdots n^n$$
$$K\left(\frac{1}{2}\right)=\frac{A^{3/2}}{2^{1/24}e^{1/8}}$$

Theorem 3.41.

$$\ln G(x)+\ln K(x)=(x-1)\ln\Gamma(x)$$

Theorem 3.42.

$$K(x) = \frac{\Gamma_{1,1}(x)}{\Gamma_{1,1}(0)}$$

3.7 多重ガンマ関数

Definition 3.43.

$$\Gamma_r(x;\boldsymbol{\alpha}) := \exp\left(\zeta_r'(0,x;\boldsymbol{\alpha})\right)$$

3.8 多重三角関数

Definition 3.44. $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_r$ として,

$$S_r(x;\boldsymbol{\alpha}) := \Gamma_r(x;\boldsymbol{\alpha})^{-1}\Gamma_r(|\boldsymbol{\alpha}|-x;\boldsymbol{\alpha})^{(-1)^r}$$

と定義する.

3.9 Hadamard のガンマ関数

Definition 3.45.

$$U = \frac{2^{x/2}}{\Gamma\left(1-\frac{x}{2}\right)}, \quad V = \frac{2^{x/2}}{\Gamma\left(\frac{1-x}{2}\right)}$$

として,

$$H(x) := \sqrt{\pi}\left(V\frac{dU}{dx} - U\frac{dV}{dx}\right)$$

と定義する.

Theorem 3.46 (特殊値)**.**

$$\begin{aligned}
&H(0) = \ln 2\\
&H(n+1) = n!\\
&H\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\\
&H(-n) = \frac{(-1)^n}{n!}\left(\ln 2 + \sum_{k=1}^{n}\frac{(-1)^k}{k}\right)
\end{aligned}$$

Theorem 3.47.

$$H(x) = xH(x) + \frac{1}{\Gamma(1-x)}$$

Theorem 3.48 (相反公式)**.**

$$\frac{H(x)}{\Gamma(x)} + \frac{H(1-x)}{\Gamma(1-x)} = 1$$

4 ゼータ関数系

4.1 Riemann ゼータ関数

Definition 4.1.

$$\zeta(s) = \sum_{0<n} \frac{1}{n^s}, \quad (\Re s > 1)$$

一般の複素数 $s \neq 1$ に対しては解析接続して定義される.

Theorem 4.2 (特殊値 I)**.** 正整数 n に対し,

$$\begin{aligned}
\zeta(2) &= \frac{\pi^2}{6} \\
\zeta(4) &= \frac{\pi^4}{90} \\
\zeta(6) &= \frac{\pi^6}{945} \\
\zeta(8) &= \frac{\pi^8}{9450} \\
\zeta(10) &= \frac{\pi^{10}}{93555} \\
\zeta(12) &= \frac{691\pi^{12}}{638512875} \\
\zeta(14) &= \frac{2\pi^{14}}{18243225} \\
\zeta(2n) &= (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}
\end{aligned}$$

Theorem 4.3 (特殊値 II)**.** 正整数 n に対し,

$$\begin{aligned}
\zeta(0) &= -\frac{1}{2} \\
\zeta(-1) &= -\frac{1}{12} \\
\zeta(-3) &= \frac{1}{120} \\
\zeta(-5) &= -\frac{1}{252} \\
\zeta(-7) &= \frac{1}{240} \\
\zeta(-9) &= -\frac{1}{132} \\
\zeta(-11) &= \frac{691}{32760} \\
\zeta(-13) &= -\frac{1}{12} \\
\zeta(-2n) &= 0 \\
\zeta(-n) &= -\frac{B_{n+1}}{n+1}
\end{aligned}$$

Theorem 4.4 (特殊値 III)**.** 正整数 n に対し,

$$\begin{aligned}
\zeta'(0) &= -\frac{1}{2}\ln(2\pi) \\
\zeta'(-2) &= -\frac{1}{4\pi^2}\zeta(3) \\
\zeta'(-4) &= \frac{3}{4\pi^4}\zeta(5) \\
\zeta'(-6) &= -\frac{45}{8\pi^6}\zeta(7) \\
\zeta'(-8) &= \frac{315}{4\pi^8}\zeta(9) \\
\zeta'(-2n) &= (-1)^n\frac{(2n)!}{2(2\pi)^{2n}}\zeta(2n+1)
\end{aligned}$$

Theorem 4.5 (Euler 積表示)**.**

$$\zeta(s) = \prod_{p:\text{prime}} \frac{1}{1-p^{-s}}$$

Theorem 4.6 (積分表示)**.**

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx$$
$$\zeta(s) = s \int_0^\infty \frac{1}{x^{s+1}} \left(\frac{1}{2} - x + \lfloor x \rfloor \right) dx$$

Theorem 4.7 (関数等式)**.**

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s)$$
$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos \frac{\pi s}{2} \Gamma(s) \zeta(s)$$

4.2 Dirichlet イータ関数

Definition 4.8.

$$\eta(s) := \sum_{0<n} \frac{(-1)^{n-1}}{n^s}, \quad \Re s > 0$$

ゼータ関数と次のような関係にある.

$$\eta(s) = (1 - 2^{1-s}) \zeta(s)$$

Theorem 4.9 (特殊値)**.**

$$\begin{aligned}
\eta(0) &= \frac{1}{2} \\
\eta(1) &= \ln 2 \\
\eta(2) &= \frac{\pi^2}{12} \\
\eta(4) &= \frac{7\pi^4}{720} \\
\eta(6) &= \frac{31\pi^6}{30240} \\
\eta(8) &= \frac{127\pi^8}{1209600} \\
\eta(10) &= \frac{73\pi^{10}}{6842880} \\
\eta(2n) &= (-1)^{n-1}\frac{(2^{2n-1}-1)B_{2n}\pi^{2n}}{(2n)!} \\
\eta'(0) &= \frac{1}{2}\ln\frac{\pi}{2} \\
\eta'(1) &= \gamma\ln 2 - \frac{\ln^2 2}{2}
\end{aligned}$$

4.3 Dirichlet ベータ関数

Definition 4.10.

$$\beta(s) := \sum_{0\le n}\frac{(-1)^n}{(2n+1)^s}, \quad \Re s > 0$$

一般の複素数に対しては解析接続して定義される.

Theorem 4.11 (特殊値)**.** 非負整数 n に対し,

$$
\begin{aligned}
\beta(0) &= \frac{1}{2} \\
\beta(1) &= \frac{\pi}{4} \\
\beta(3) &= \frac{\pi^3}{32} \\
\beta(5) &= \frac{5\pi^5}{1536} \\
\beta(7) &= \frac{61\pi^7}{184320} \\
\beta(2n+1) &= (-1)^n \frac{E_{2n}}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1} \\
\beta(-n) &= \frac{E_n}{2} \\
\beta'(1) &= \frac{\pi}{4}(\gamma - \ln \pi) + \pi \ln \Gamma\left(\frac{3}{4}\right)
\end{aligned}
$$

Theorem 4.12 (積分表示)**.**

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-2x}}\, dx$$

Theorem 4.13 (関数等式)**.**

$$
\begin{aligned}
\beta(s) &= \left(\frac{\pi}{2}\right)^{s-1} \cos\frac{\pi s}{2} \Gamma(1-s)\beta(1-s) \\
\beta(1-s) &= \left(\frac{\pi}{2}\right)^{-s} \sin\frac{\pi s}{2} \Gamma(s)\beta(s)
\end{aligned}
$$

4.4 Dirichlet L 関数

Definition 4.14 (Dirichlet 指標)**.** 任意の整数 a, b に対し, $\mathbb{Z}$ 上の複素数値関数で,

1. $\chi(a)\chi(b) = \chi(ab)$
2. $\chi(a+n) = \chi(a)$
3. $\chi(a) \neq 0$ ならば, a と n は互いに素.

を満たすものを, n を法とする Dirichlet 指標という.

Definition 4.15. Dirichlet 指標に対し,

$$L(s, \chi) := \sum_{0<n} \frac{\chi(n)}{n^s}$$

と定義する.

Theorem 4.16 (関数等式)**.**

$$\left(\frac{\pi}{n}\right)^{-s/2}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)=\frac{G(\chi)}{i^a\sqrt{n}}\left(\frac{\pi}{n}\right)^{-(1-s)/2}\Gamma\left(\frac{1-s+a}{2}\right)L(1-s,\overline{\chi})$$

ここで, a は指標の偶奇によって, $0,1$ のどちらかをとり,

$$G(\chi):=\sum_{k=0}^{n-1}\chi(k)e^{2\pi ik/n}$$

である.

4.5 Hurwitz ゼータ関数

Definition 4.17.

$$\zeta(s,x):=\sum_{0\leq n}\frac{1}{(n+x)^s}$$

一般の複素数 $s\neq 1$ に対しては解析接続して定義される.

Theorem 4.18 (特殊値)**.** 正整数 n に対して,

$$\zeta(0,x)=\frac{1}{2}-x$$
$$\zeta(-n,x)=-\frac{B_{n+1}(x)}{n+1}$$

Theorem 4.19 (Distribution relation)**.**

$$\sum_{k=0}^{n-1}\zeta\left(s,x+\frac{k}{n}\right)=n^s\zeta(s,nx)$$

Theorem 4.20 (積分表示)**.**

$$\zeta(s,a)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx$$

Theorem 4.21 (関数等式)**.** 正整数 $0<a\leq b$ に対し,

$$\zeta\left(1-s,\frac{a}{b}\right)=\frac{2\Gamma(s)}{(2\pi b)^s}\sum_{k=1}^{b}\cos\left(\frac{\pi s}{2}-\frac{2\pi ak}{b}\right)\zeta\left(s,\frac{k}{b}\right)$$

Theorem 4.22 (Hurwitz's formula)**.**

$$\zeta(1-s,x) = \frac{2\Gamma(s)}{(2\pi)^s} \sum_{0<n} \frac{1}{n^s} \cos\left(\frac{\pi s}{2} - 2\pi n x\right)$$

4.6 Barnes 多重ゼータ関数

Definition 4.23.

$$\zeta_r(s,x;\boldsymbol{\alpha}) := \sum_{0\leq n_1,\ldots n_r} \frac{1}{(x+\boldsymbol{n}\cdot\boldsymbol{\alpha})^s}$$

Theorem 4.24 (積分表示)**.**

$$\zeta_r(s,x;\boldsymbol{\alpha}) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-xt}}{\prod_{i=1}^r (1-e^{-\alpha_i t})}\,dt, \quad (\Re s > n)$$

4.7 Riemann-Siegel 関数

Definition 4.25.

$$Z(x) := e^{i\theta(x)}\zeta\left(\frac{1}{2}+ix\right)$$
$$\theta(x) := \Im \ln\Gamma\left(\frac{1}{4}+i\frac{x}{2}\right) - \frac{x}{2}\ln\pi$$

4.8 Ramanujan L 関数

Definition 4.26.

$$L_\tau(s) := \sum_{0<n} \frac{\tau(n)}{n^s}$$

Theorem 4.27 (Euler 積表示)**.**

$$L_\tau(s) = \prod_{p:\mathrm{prime}} \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}$$

Theorem 4.28 (関数等式)**.**

$$(2\pi)^{-s}\Gamma(s)L_\tau(s) = (2\pi)^{s-12}\Gamma(12-s)L_\tau(12-s)$$

5　多重対数関数系

5.1　多重対数関数

Definition 5.1.

$$\mathrm{Li}_s(x) = \sum_{0<n} \frac{x^n}{n^s}, \quad |x| \leq 1$$

一般の複素数に対しては解析接続して定義される.

Theorem 5.2 (Li$_2$ の特殊値)**.**

$$\rho = \frac{\sqrt{5}-1}{2}$$

とする.

$$\begin{aligned}
\mathrm{Li}_2(1) &= \frac{\pi^2}{6} \\
\mathrm{Li}_2(-1) &= -\frac{\pi^2}{12} \\
\mathrm{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{\ln^2 2}{2} \\
\mathrm{Li}_2(\rho) &= \frac{\pi^2}{10} - \ln^2 \rho \\
\mathrm{Li}_2(-\rho) &= \frac{\ln^2 \rho}{2} - \frac{\pi^2}{15} \\
\mathrm{Li}_2(\rho^2) &= \frac{\pi^2}{15} - \ln^2 \rho \\
\mathrm{Li}_2(i) &= -\frac{\pi^2}{48} + i\beta(2)
\end{aligned}$$

Theorem 5.3 (Li$_3$ の特殊値)**.**

$$\rho = \frac{\sqrt{5}-1}{2}$$

とする.

$$
\begin{aligned}
\mathrm{Li}_3(1) &= \zeta(3) \\
\mathrm{Li}_3(-1) &= -\frac{3}{4}\zeta(3) \\
\mathrm{Li}_3\left(\frac{1}{2}\right) &= \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\ln 2 + \frac{1}{6}\ln^3 2 \\
\mathrm{Li}_3(\rho^2) &= \frac{4}{5}\zeta(3) + \frac{2\pi^2}{15}\ln\rho - \frac{2}{3}\ln^3\rho
\end{aligned}
$$

Theorem 5.4 (s に関する特殊値)**.**

$$
\begin{aligned}
\mathrm{Li}_1(x) &= -\ln(1-x) \\
\mathrm{Li}_0(x) &= \frac{x}{1-x} \\
\mathrm{Li}_{-1}(x) &= \frac{x}{(1-x)^2} \\
\mathrm{Li}_{-2}(x) &= \frac{x(1+x)}{(1-x)^3} \\
\mathrm{Li}_{-3}(x) &= \frac{x(1+4x+x^2)}{(1-x)^4} \\
\mathrm{Li}_{-4}(x) &= \frac{x(1+x)(1+10x+x^2)}{(1-x)^5}
\end{aligned}
$$

Theorem 5.5 (Li_2 の関係式)**.**

$$
\begin{aligned}
&\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x) \\
&\mathrm{Li}_2(x) + \mathrm{Li}_2\left(-\frac{x}{1-x}\right) = -\frac{1}{2}\ln^2(1-x) \\
&\mathrm{Li}_2(-x) + \mathrm{Li}_2\left(-\frac{1}{x}\right) = -\frac{\pi^2}{6} - \frac{\ln^2 x}{2} \\
&\mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2\left(\frac{x}{1-y}\frac{y}{1-x}\right) = \mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_1(x)\mathrm{Li}_1(y) \\
&\mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2(xy) \\
&\quad = \mathrm{Li}_2\left(\frac{x(1-y)}{1-xy}\right) + \mathrm{Li}_2\left(\frac{y(1-x)}{1-xy}\right) + \mathrm{Li}_1\left(\frac{x(1-y)}{1-xy}\right)\mathrm{Li}_1\left(\frac{y(1-x)}{1-xy}\right)
\end{aligned}
$$

Theorem 5.6 (Li_3 の関係式)**.**

$$\mathrm{Li}_3(-x) + \mathrm{Li}_3\left(-\frac{1}{x}\right) = -\frac{1}{6}\ln^3 x - \frac{\pi^2}{6}\ln x$$
$$\mathrm{Li}_3(x) + \mathrm{Li}_3(1-x) + \mathrm{Li}_3\left(1-\frac{1}{x}\right) = \zeta(3) + \frac{1}{6}\ln^3 x + \frac{\pi^2}{6}\ln x - \frac{1}{2}\ln^2 x\ln(1-x)$$

Theorem 5.7 (乗法定理)**.**

$$\sum_{k=0}^{n-1}\mathrm{Li}_s(xe^{2\pi ik/n}) = n^{1-s}\mathrm{Li}_s(x^n)$$

5.2 Roger's dilogarithm

Definition 5.8.

$$L(x) := \frac{6}{\pi^2}\left(\mathrm{Li}_2(x) + \frac{1}{2}\ln x\ln(1-x)\right)$$

Theorem 5.9 (特殊値 I)**.**

$$L(0) = 0$$
$$L(1) = 1$$
$$L\left(\frac{1}{2}\right) = \frac{1}{2}$$
$$2 - 6L\left(\frac{1}{4}\right) - 2L\left(\frac{1}{8}\right) + L\left(\frac{1}{64}\right) = 0$$
$$2 - 6L\left(\frac{1}{3}\right) + L\left(\frac{1}{9}\right) = 0$$

Theorem 5.10 (特殊値 II)**.**

$$\rho = \frac{\sqrt{5}-1}{2}$$

とする.

$$
\begin{aligned}
&L(\rho) = \frac{3}{5} \\
&L(\rho^2) = \frac{2}{5} \\
&L(\rho^6) - 4L(\rho^3) - 3L(\rho^2) + 6L(\rho) = \frac{7}{5} \\
&L(\rho^{12}) - 2L(\rho^6) - 3L(\rho^4) - 4L(\rho^3) + 6L(\rho^2) = \frac{3}{5} \\
&L(\rho^{20}) - 2L(\rho^{10}) - 15L(\rho^4) + 10L(\rho^2) = \frac{6}{5} \\
&L(\sqrt{\rho}) + L\left(\frac{1}{1+\sqrt{\rho}}\right) = \frac{13}{11}
\end{aligned}
$$

Theorem 5.11 (特殊値 III)**.**

$$
\begin{aligned}
&\alpha = \frac{1}{2\cos\frac{2\pi}{7}} \\
&\beta = \frac{1}{2\cos\frac{\pi}{7}} \\
&\gamma = 2\cos\frac{3\pi}{7} \\
&\delta = \frac{\sqrt{3+2\sqrt{5}}-1}{2} \\
&a = \frac{1}{2\cos\frac{\pi}{9}} \\
&b = \frac{1}{2\cos\frac{2\pi}{9}} \\
&c = 2\cos\frac{4\pi}{9} \\
&d = 2\sqrt{3}\cos\frac{5\pi}{18} - 2 \\
&e = 2\sqrt{3}\cos\frac{11\pi}{18} + 2 \\
&f = 2\sqrt{3}\cos\frac{7\pi}{18} - 1
\end{aligned}
$$

とする.

$$
\begin{aligned}
&L(\alpha) - L(\alpha^2) = \frac{1}{7} \\
&2L(\beta) + L(\beta^2) = \frac{10}{7} \\
&2L(\gamma) + L(\gamma^2) = \frac{8}{7} \\
&5L(\delta^3) - 5L(\delta) + 1 = 0 \\
&L(\delta^{12}) - 2L(\delta^6) - 6L(\delta^4) + 4L(\delta^3) + 3L(\delta^2) + 4L(\delta) - 4 = 0 \\
&3L(a^3) - 9L(a^2) - 9L(a) + 7 = 0 \\
&3L(b^6) - 6L(b^3) - 27L(b^2) + 18L(b) + 2 = 0 \\
&3L(c^6) - 6L(c^3) - 27L(c^2) + 18L(c) - 2 = 0 \\
&2L(d^3) - 2L(d^2) - 11L(d) + 3 = 0 \\
&2L(e^6) - 4L(e^3) - 15L(e^2) + 22L(e) - 6 = 0 \\
&2L(f^6) - 4L(f^3) - 15L(f^2) + 22L(f) - 4 = 0
\end{aligned}
$$

Theorem 5.12 (関係式)**.**

$$
\begin{aligned}
&L(x) + L(1-x) = 1 \\
&L(x) + L(y) - L(xy) = L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right) \\
&L(x) + L\left(\frac{1}{x}\right) = 2 \\
&1 + L(x) = L\left(\frac{1}{1-x}\right) \\
&L(x^2) = 2L(x) - 2L\left(\frac{x}{1+x}\right)
\end{aligned}
$$

5.3 Legendre のカイ関数

Definition 5.13.

$$\chi_s(x) = \sum_{0 \le n} \frac{x^{2n+1}}{(2n+1)^s}$$

多重対数関数と以下の関係にある.

$$\chi_s(x) = \frac{1}{2}(\mathrm{Li}_s(x) - \mathrm{Li}_s(-x))$$

Theorem 5.14 (χ_2 の特殊値)**.**

$$\rho = \frac{\sqrt{5}-1}{2}$$

とする.

$$\begin{aligned}
\chi_2(1) &= \frac{\pi^2}{8} \\
\chi_2(\sqrt{2}-1) &= \frac{\pi^2}{16} - \frac{\ln^2(\sqrt{2}-1)}{4} \\
\chi_2(\rho) &= \frac{\pi^2}{12} - \frac{3\ln^2\rho}{4} \\
\chi_2(\rho^3) &= \frac{\pi^2}{24} - \frac{3\ln^2\rho}{4} \\
\chi_2\left(\frac{1}{\sqrt{2}}\right) &= \mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right) + \frac{\ln^2 2}{8} - \frac{\pi^2}{48} \\
\chi_2\left((\sqrt{2}-1)^2\right) &= \frac{7\pi^2}{48} + \frac{1}{2}\ln 2\ln(\sqrt{2}-1) - \frac{\ln^2 2}{8} - \mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right)
\end{aligned}$$

Theorem 5.15 (χ_2 の関係式)**.**

$$\chi_2(x) + \chi_2\left(\frac{1-x}{1+x}\right) = \frac{\pi^2}{8} - \frac{1}{2}\ln x \ln\left(\frac{1-x}{1+x}\right)$$

5.4 Inverse tangent integral

Definition 5.16.

$$\mathrm{Ti}_s(x) := \sum_{0\le n} \frac{(-1)^n}{(2n+1)^s} x^{2n+1}$$

Theorem 5.17 (特殊値)**.**

$$\mathrm{Ti}_s(1) = \beta(s)$$

Theorem 5.18.

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\arctan t}{t}\,dt$$

Theorem 5.19 (Fourier 級数)**.**

$$\mathrm{Ti}_2(\tan x) = x\ln\tan x + \sum_{0\le n} \frac{\sin(4n+2)x}{(2n+1)^2}$$

5.5 Clausen 関数

Definition 5.20. 正整数 r に対し,

$$
\begin{aligned}
\mathrm{Cl}_{2r}(x) &= \sum_{0<n} \frac{\sin nx}{n^{2r}} \\
\mathrm{Cl}_{2r-1}(x) &= \sum_{0<n} \frac{\cos nx}{n^{2r-1}} \\
\mathrm{Sl}_{2r}(x) &= \sum_{0<n} \frac{\cos nx}{n^{2r}} \\
\mathrm{Sl}_{2r-1}(x) &= \sum_{0<n} \frac{\sin nx}{n^{2r-1}}
\end{aligned}
$$

と定義する.

Theorem 5.21 (Sl の特殊値)**.** $0 < x < 2\pi$ としたとき,

$$
\begin{aligned}
\mathrm{Sl}_1(x) &= \frac{\pi}{2} - \frac{x}{2} \\
\mathrm{Sl}_2(x) &= \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \\
\mathrm{Sl}_3(x) &= \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} \\
\mathrm{Sl}_4(x) &= \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48} \\
\mathrm{Sl}_r(x) &= \frac{(-1)^{\lfloor r/2 \rfloor - 1}(2\pi)^r}{2r!} B_r\left(\frac{x}{2\pi}\right)
\end{aligned}
$$

Theorem 5.22 (Cl の特殊値)**.**

$$
\begin{aligned}
&\mathrm{Cl}_{2r}(\pi n) = 0 \\
&\mathrm{Cl}_{2r}\left(\frac{\pi}{2}\right) = \beta(2r) \\
&\mathrm{Cl}_{2r-1}(0) = \zeta(2r-1) \\
&\mathrm{Cl}_{2r-1}(\pi) = -\eta(2r-1) \\
&\mathrm{Cl}_{2r-1}\left(\frac{\pi}{2}\right) = -2^{1-2r}\eta(2r-1)
\end{aligned}
$$

Theorem 5.23 (Duplication formula)**.**

$$
\mathrm{Cl}_r(2x) = 2^{r-1}(\mathrm{Cl}_r(x) + (-1)^{r-1}\mathrm{Cl}(\pi - x))
$$

Theorem 5.24.

$$\mathrm{Cl}_{2r}(x) = \Im(\mathrm{Li}_{2r}(e^{ix}))$$
$$\mathrm{Cl}_{2r-1}(x) = \Re(\mathrm{Li}_{2r-1}(e^{ix}))$$

Theorem 5.25.

$$\int_0^x \ln\sin x\,dx = -\frac{1}{2}\mathrm{Cl}_2(2x) - x\ln 2$$
$$\int_0^x \ln\cos x\,dx = \frac{1}{2}\mathrm{Cl}_2(\pi - 2x) - x\ln 2$$
$$\int_0^x \ln\tan x\,dx = -\frac{1}{2}\mathrm{Cl}_2(2x) - \frac{1}{2}\mathrm{Cl}_2(\pi - 2x)$$

5.6 Lerch Transcendent

Definition 5.26.

$$\Phi(z,s,a) := \sum_{0\le n} \frac{z^n}{(n+a)^s}$$

Theorem 5.27.

$$\Phi(1,s,a) = \zeta(s,a)$$
$$\Phi(z,s,1) = \frac{\mathrm{Li}_s(z)}{z}$$
$$\Phi\left(z^2, s, \frac{1}{2}\right) = 2^s z\chi_s(z)$$

Theorem 5.28 (積分表示)**.**

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-ax}}{1-ze^{-x}}\,dt$$

6 多重ゼータ値系

6.1 多重ゼータ値

Definition 6.1. $\boldsymbol{k} = (k_1,\ldots,k_a)$ に対し,

$$\zeta(\boldsymbol{k}) := \sum_{0<n_1<\cdots<n_a} \frac{1}{\boldsymbol{n}^{\boldsymbol{k}}}$$

と定義する. ここで, $\boldsymbol{n}^{\boldsymbol{k}} = n_1^{k_1} \cdots n_a^{k_a}$ である. a を $\mathrm{dep}(\boldsymbol{k})$ と表し, $\mathrm{wt}(\boldsymbol{k}) := k_1 + \cdots + k_a$ とする.

Theorem 6.2 (特殊値)**.**

$$
\begin{aligned}
\zeta(\{2\}^n) &= \frac{\pi^{2n}}{(2n+1)!} \\
\zeta(\{4\}^n) &= \frac{2^{2n+1}\pi^{4n}}{(4n+2)!} \\
\zeta(\{6\}^n) &= \frac{6(2\pi)^{6n}}{(6n+3)!} \\
\zeta(\{8\}^n) &= \frac{8(2\pi)^{8n}}{(8n+4)!}\left(\left(1+\frac{1}{\sqrt{2}}\right)^{4n+2} + \left(1-\frac{1}{\sqrt{2}}\right)^{4n+2}\right) \\
\zeta(\{10\}^n) &= \frac{10(2\pi)^{10n}(1+L_{10n+5})}{(10n+5)!} \\
\zeta(\{12\}^n) &= \frac{12(2\pi)^{12n}}{(12n+6)!}\left(\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{12n+6} + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{12n+6} + 2^{6n+3}\right) \\
\zeta(\{1,3\}^n) &= \frac{2\pi^{4n}}{(4n+2)!}
\end{aligned}
$$

Theorem 6.3 (関係式)**.**

$$
\begin{aligned}
\zeta(3) &= \zeta(1,2) \\
\zeta(4) &= \zeta(1,1,2) \\
\zeta(1,3) &= \frac{1}{4}\zeta(4) \\
\zeta(2,2) &= \frac{3}{4}\zeta(4) \\
\zeta(5) &= \zeta(1,1,1,2) \\
\zeta(1,4) &= \zeta(1,1,3) \\
\zeta(2,3) &= \zeta(1,2,2) \\
\zeta(3,2) &= \zeta(2,1,2) \\
\zeta(2,3) &= \frac{1}{2}\zeta(5) - 3\zeta(1,4) \\
\zeta(3,2) &= \frac{1}{2}\zeta(5) + 2\zeta(1,4)
\end{aligned}
$$

$$
\begin{aligned}
\zeta(6) &= \zeta(1,1,1,1,2) \\
\zeta(1,5) &= \zeta(1,1,1,3) \\
\zeta(2,4) &= \zeta(1,1,2,2) \\
\zeta(3,3) &= \zeta(1,2,1,2) \\
\zeta(4,2) &= \zeta(2,1,1,2) \\
\zeta(1,3,2) &= \zeta(2,1,3) \\
\zeta(2,4) &= \frac{1}{6}\zeta(6) - 2\zeta(1,5) \\
\zeta(3,3) &= \frac{1}{4}\zeta(6) - \zeta(1,5) \\
\zeta(4,2) &= \frac{7}{12}\zeta(6) + 2\zeta(1,5) \\
\zeta(1,1,4) &= 2\zeta(1,5) - \frac{1}{16}\zeta(6) \\
\zeta(1,2,3) &= \frac{13}{48}\zeta(6) - 6\zeta(1,5) \\
\zeta(1,3,2) &= 3\zeta(1,5) - \frac{1}{24}\zeta(6) \\
\zeta(2,2,2) &= \frac{3}{16}\zeta(6) \\
\zeta(3,1,2) &= \frac{11}{16}\zeta(6) - 2\zeta(1,5)
\end{aligned}
$$

Theorem 6.4 (和公式)**.**

$$
\sum_{\substack{\mathrm{dep}(\boldsymbol{k})=a \\ \mathrm{wt}(\boldsymbol{k})=k}} \zeta(\boldsymbol{k}) = \zeta(k)
$$

Theorem 6.5 (双対性)**.**

$$
\boldsymbol{k} = (\{1\}^{a_1-1}, b_1, \ldots, \{1\}^{a_r-1}, b_r)
$$

としたとき, その双対インデックスを,

$$
\boldsymbol{k}^\dagger = (\{1\}^{b_r-1}, a_r, \ldots, \{1\}^{b_1-1}, a_1)
$$

と定義する. このとき,

$$
\zeta(\boldsymbol{k}) = \zeta(\boldsymbol{k}^\dagger)
$$

Theorem 6.6 (大野関係式)**.** インデックスの和を $\boldsymbol{k}+\boldsymbol{e} = (k_1+e_1, \ldots, k_a+e_a)$ とする.

$$
O_h(\boldsymbol{k}) = \sum_{\substack{0\le e_i \\ e_1+\cdots+e_a=h}} \zeta(\boldsymbol{k}+\boldsymbol{e})
$$

としたとき,

$$O_h(\boldsymbol{k}) = O_h(\boldsymbol{k}^\dagger)$$

Theorem 6.7.

$$\sum_{0\le n} \zeta(\{r\}^n) x^{rn} = \exp\left(\sum_{0<n} \frac{(-1)^{n-1}\zeta(rn)}{n} x^{rn}\right)$$

Theorem 6.8.

$$\sum_{0<n,m} \zeta(\{1\}^{n-1}, m+1) x^n y^m = 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}$$

6.2 等号付き多重ゼータ値

Definition 6.9.

$$\zeta^*(\boldsymbol{k}) := \sum_{0<n_1\le\cdots\le n_a} \frac{1}{\boldsymbol{n}^{\boldsymbol{k}}}$$

Theorem 6.10 (特殊値)**.**

$$\zeta^*(\{2\}^n) = 2(1-2^{1-2n})\zeta(2n)$$
$$\zeta^*(1,\{2\}^n) = 2\zeta(2n+1)$$
$$\zeta^*(\{1\}^{n-1},2) = n\zeta(n+1)$$

Theorem 6.11 (和公式)**.**

$$\sum_{\substack{\mathrm{dep}(\boldsymbol{k})=r \\ \mathrm{wt}(\boldsymbol{k})=k}} \zeta^*(\boldsymbol{k}) = \binom{k-1}{r-1}\zeta^*(k)$$

Theorem 6.12 (Weighted sum formula)**.**

$$\sum_{k=2}^{r-1} 2^k \zeta(r-k,k) = (r+1)\zeta(r)$$

6.3 交代多重ゼータ値

Definition 6.13. $\varepsilon_i = \pm 1$ として,

$$\zeta(\boldsymbol{k};\boldsymbol{\varepsilon}) := \sum_{0<n_1<\cdots<n_a} \frac{\boldsymbol{\varepsilon}^{\boldsymbol{n}}}{\boldsymbol{n}^{\boldsymbol{k}}}$$

$\varepsilon_i = -1$ のとき, k_i を $\overline{k_i}$ と書いて, 1 つの交代インデックスで,

$$\zeta(\boldsymbol{k}) := \zeta(\boldsymbol{k}; \boldsymbol{\varepsilon})$$

と表す.

Theorem 6.14 (特殊値)**.**

$$\begin{aligned}
\zeta(\{\overline{2}\}^n) &= \frac{(-1)^{n(n+1)/2}\pi^{2n}}{2^n(2n+1)!} \\
\zeta(\{\overline{4}\}^n) &= \frac{(-1)^{n(n+1)/2}\pi^{4n}}{(4n+2)!}\left((1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}\right) \\
\zeta(\{\overline{6}\}^n) &= \frac{3\pi^{6n}}{2(6n+3)!} \\
&\cdot \left(1 + (-1)^{n(n+1)/2} 2^{3n+1} \left(\left(\frac{1+\sqrt{3}}{2}\right)^{6n+3} + \left(\frac{1-\sqrt{3}}{2}\right)^{6n+3} - 1\right)\right) \\
\zeta(\{1, \overline{2}\}^n) &= 8^{-n}\zeta(\{3\}^n)
\end{aligned}$$

Theorem 6.15 (weight 1,2 の特殊値)**.**

$$\begin{aligned}
\zeta(\overline{1}) &= -\ln 2 \\
\zeta(\overline{2}) &= -\frac{\pi^2}{12} \\
\zeta(1, \overline{1}) &= \frac{\ln^2 2}{2} \\
\zeta(\overline{1}, \overline{1}) &= \frac{\ln^2 2}{2} - \frac{\pi^2}{12}
\end{aligned}$$

Theorem 6.16 (weight 3 の特殊値)**.**

$$
\begin{aligned}
\zeta(\overline{3}) &= -\frac{3}{4}\zeta(3) \\
\zeta(1,\overline{2}) &= \frac{1}{8}\zeta(3) \\
\zeta(\overline{1},2) &= \zeta(3) - \frac{\pi^2}{4}\ln 2 \\
\zeta(\overline{1},\overline{2}) &= \frac{\pi^2}{4}\ln 2 - \frac{13}{8}\zeta(3) \\
\zeta(2,\overline{1}) &= \frac{\pi^2}{12}\ln 2 - \frac{1}{4}\zeta(3) \\
\zeta(\overline{2},\overline{1}) &= \frac{5}{8}\zeta(3) - \frac{\pi^2}{6}\ln 2 \\
\zeta(1,1,\overline{1}) &= -\frac{1}{6}\ln^3 2 \\
\zeta(1,\overline{1},\overline{1}) &= -\frac{7}{8}\zeta(3) + \frac{\pi^2}{12}\ln 2 - \frac{1}{6}\ln^3 2 \\
\zeta(\overline{1},1,\overline{1}) &= \frac{\zeta(3)}{8} - \frac{\ln^3 2}{6} \\
\zeta(\overline{1},\overline{1},\overline{1}) &= -\frac{1}{4}\zeta(3) + \frac{\pi^2}{12}\ln 2 - \frac{1}{6}\ln^3 2
\end{aligned}
$$

Theorem 6.17 (weight 4 の特殊値)**.**

$$
\begin{aligned}
\zeta(\overline{4}) &= -\frac{7}{720}\pi^4 \\
\zeta(\overline{2},\overline{2}) &= -\frac{3}{1440}\pi^4 \\
\zeta(\overline{1},3) &= \frac{19}{1440}\pi^4 - \frac{7}{4}\zeta(3)\ln 2 \\
\zeta(3,\overline{1}) &= \frac{3}{4}\zeta(3)\ln 2 - \frac{5}{1440}\pi^4 \\
\zeta(1,1,1,\overline{1}) &= \frac{1}{24}\ln^4 2
\end{aligned}
$$

6.4 Multiple T-value

Definition 6.18.

$$T(\boldsymbol{k}) := 2^a \sum_{0<n_1<\cdots<n_a} \frac{1}{(2n_1-1)^{k_1}\cdots(2n_a-r)^{k_a}}$$
$$t(\boldsymbol{k}) := \sum_{0<n_1<\cdots<n_a} \frac{1}{(2n_1-1)^{k_1}\cdots(2n_a-1)^{k_a}}$$
$$t^*(\boldsymbol{k}) := \sum_{0<n_1\leq\cdots\leq n_a} \frac{1}{(2n_1-1)^{k_1}\cdots(2n_a-1)^{k_a}}$$

Theorem 6.19 ($T(\boldsymbol{k})$ の特殊値)**.**

$$T(2) = \frac{\pi^2}{4}$$
$$T(3) = \frac{7}{4}\zeta(3)$$
$$T(4) = \frac{\pi^4}{48}$$
$$T(r) = 2(1-2^{-r})\zeta(r)$$

Theorem 6.20 ($t(\boldsymbol{k})$ の特殊値)**.**

$$
\begin{aligned}
t(2) &= \frac{\pi^2}{8} \\
t(3) &= \frac{7}{8}\zeta(3) \\
t(4) &= \frac{\pi^4}{96} \\
t(r) &= (1-2^{-r})\zeta(r) \\
t(1,2) &= -\frac{7}{16}\zeta(3) + \frac{1}{8}\pi^2 \ln 2 \\
t(2,2) &= \frac{\pi^4}{384} \\
t(1,1,2) &= \frac{11}{5760}\pi^4 - \frac{1}{16}7\zeta(3)\ln 2 + \frac{1}{16}\pi^2 \ln^2 2 + \frac{1}{4}\zeta(1,\overline{3}) \\
t(1,4) &= -\frac{31}{64}\zeta(5) - \frac{1}{448}\pi^2\zeta(3) + \frac{1}{96}\pi^4 \ln 2 \\
t(2,3) &= -\frac{31}{64}\zeta(5) + \frac{3}{448}\pi^2\zeta(3) \\
t(3,2) &= -\frac{31}{64}\zeta(5) + \frac{1}{112}\pi^2\zeta(3) \\
t(1,\overline{1}) &= \frac{\beta(2)}{2} - \frac{\pi \ln 2}{8} \\
t(\{2\}^n) &= \frac{\pi^{2n}}{2^{2n}(2n)!} \\
t(\{4\}^n) &= \frac{\pi^{4n}}{2^{2n}(4n)!} \\
t(\{6\}^n) &= \frac{3\pi^{6n}}{4(6n)!} \\
t(\{\overline{1}\}^n) &= \frac{(-1)^{n(n+1)/2}\pi^n}{2^{2n}n!} \\
t(\{\overline{3}\}^n) &= \frac{(-1)^{n(n+1)/2}3\pi^{3n}}{2^{3n+1}(3n)!} \\
t^*(\{2\}^n) &= \frac{(-1)^n E_{2n}\pi^{2n}}{2^{2n}(2n)!}
\end{aligned}
$$

Theorem 6.21 (双対性)**.**

$$T(\boldsymbol{k}) = T(\boldsymbol{k}^\dagger)$$

Theorem 6.22.

$$\sum_{0<n,m} T(\{1\}^{n-1}, m+1)x^n y^m = 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} {}_2F_1\left[\begin{matrix}-x, -y\\1-x-y\end{matrix}; -1\right]$$

$$\sum_{0<n,m} t(\{1\}^{n-1}, m+1)x^n y^m = {}_3F_2\left[\begin{matrix}1, \frac{1-x}{2}, \frac{1+y}{2}\\ \frac{3}{2}, \frac{3-x}{2}\end{matrix}; 1\right]$$

6.5 Multiple Polylogarithms

Definition 6.23.

$$\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{z}) := \sum_{0=n_0<n_1<\cdots<n_a} \prod_{i=1}^{a} \frac{z_i^{n_i-n_{i-1}}}{n_i^{k_i}}$$

$$\mathrm{Li}_{\boldsymbol{k}}(z) := \mathrm{Li}_{\boldsymbol{k}}(z, \ldots, z)$$

Theorem 6.24 ($z = \frac{1}{2}$ での特殊値)**.**

$$\mathrm{Li}_{1,1}\left(\frac{1}{2}\right) = \frac{\ln^2 2}{2}$$

$$\mathrm{Li}_{1,2}\left(\frac{1}{2}\right) = \frac{\zeta(3)}{8} - \frac{\ln^3 2}{6}$$

$$\mathrm{Li}_{2,1}\left(\frac{1}{2}\right) = -\frac{1}{4}\zeta(3) + \frac{\pi^2}{12}\ln 2 - \frac{1}{6}\ln^3 2$$

$$\mathrm{Li}_{3}\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\ln 2 + \frac{1}{6}\ln^3 2$$

$$\mathrm{Li}_{1,1,1}\left(\frac{1}{2}\right) = \frac{1}{6}\ln^3 2$$

Theorem 6.25.

$$\mathrm{Li}_{1,1}(x, y) = \mathrm{Li}_2\left(\frac{x-y}{1-y}\right) - \mathrm{Li}_2\left(-\frac{y}{1-y}\right) - \mathrm{Li}_2(x)$$

$$\mathrm{Li}_{\{1\}^r}(z) = \frac{(-\ln(1-z))^r}{r!}$$

Theorem 6.26. $-1 < \Re z_i < \frac{1}{2}$ とする.

$$\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{z}) = \mathrm{Li}_{\{1\}^{\mathrm{wt}(\boldsymbol{k})}}\left(\{1\}^{k_a-1}, -\frac{z_a}{1-z_a}, \ldots, \{1\}^{k_1-1}, -\frac{z_1}{1-z_1}\right)$$

Theorem 6.27 (Landen connection formula)**.**

$$\mathrm{Li}_{\boldsymbol{k}}\left(-\frac{z}{1-z}\right) = (-1)^r \sum_{\boldsymbol{k}\preceq\boldsymbol{k}'} \mathrm{Li}_{\boldsymbol{k}'}(z)$$

7　楕円関数系

7.1　テータ関数

Definition 7.1.

$$\vartheta_0(x;\tau) := \vartheta_{01}(x;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i\tau n^2+2\pi in\left(x+\frac{1}{2}\right)}$$

$$\vartheta_1(x;\tau) := -\vartheta_{11}(x;\tau) = -\sum_{n=-\infty}^{\infty} e^{\pi i\tau\left(n+\frac{1}{2}\right)^2+2\pi i\left(n+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)}$$

$$\vartheta_2(x;\tau) := \vartheta_{10}(x;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i\tau\left(n+\frac{1}{2}\right)^2+2\pi i\left(n+\frac{1}{2}\right)x}$$

$$\vartheta_3(x;\tau) := \vartheta_{00}(x;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i\tau n^2+2\pi inx}$$

また, $q = e^{\pi i\tau}$ を変数として,

$$\vartheta_0(x,q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi in\left(x+\frac{1}{2}\right)}$$

$$\vartheta_1(x,q) := -\sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{2\pi i\left(n+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)}$$

$$\vartheta_2(x,q) := \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{2\pi i\left(n+\frac{1}{2}\right)x}$$

$$\vartheta_3(x,q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi inx}$$

$\vartheta_i(x) := \vartheta_i(x;\tau), \quad \vartheta_i := \vartheta_i(0;\tau)$ と書くこともあり, $\vartheta_4 := \vartheta_0$ とする.

Theorem 7.2 (特殊値)**.**

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$C := \frac{\sqrt[4]{\pi}}{\Gamma\left(\frac{3}{4}\right)}$$

とする.

$$\varphi(e^{-\pi}) = C$$
$$\varphi(e^{-2\pi}) = \frac{\sqrt[4]{6+4\sqrt{2}}}{2}C$$
$$\varphi(e^{-3\pi}) = \frac{\sqrt[4]{3+2\sqrt{3}}}{\sqrt{3}}C$$
$$\varphi(e^{-4\pi}) = \frac{2+\sqrt[4]{8}}{4}C$$
$$\varphi(e^{-5\pi}) = \frac{\sqrt[4]{9+4\sqrt{5}}}{\sqrt{5}}C$$
$$\varphi(e^{-6\pi}) = \frac{\sqrt{1+\sqrt{2}+\sqrt{3}+\sqrt[4]{3}}}{\sqrt[8]{1728}}C$$
$$\varphi(e^{-7\pi}) = \frac{\sqrt[4]{7+4\sqrt{7}+5\sqrt[4]{28}+\sqrt[4]{1372}}}{\sqrt{7}}C$$
$$\varphi(e^{-8\pi}) = \frac{\sqrt{2+\sqrt{2}}+\sqrt[8]{128}}{4}C$$
$$\varphi(e^{-9\pi}) = \frac{1+(1+\sqrt{3})\sqrt[3]{2-\sqrt{3}}}{3}C$$

Theorem 7.3 (無限積表示)**.**

$$\vartheta_0(x,q) = \prod_{0<n}(1-q^{2n})(1-2q^{2n-1}\cos 2x + q^{4n-2})$$
$$\vartheta_1(x,q) = 2q^{1/4}\sin x\prod_{0<n}(1-q^{2n})(1-2q^{2n}\cos 2x + q^{4n})$$
$$\vartheta_2(x,q) = 2q^{1/4}\cos x\prod_{0<n}(1-q^{2n})(1+2q^{2n}\cos 2x + q^{4n})$$
$$\vartheta_3(x,q) = \prod_{0<n}(1-q^{2n})(1+2q^{2n-1}\cos 2x + q^{4n-2})$$

Theorem 7.4 (Jacobi の虚数変換式)**.**

$$\vartheta_0\left(\frac{x}{\tau};-\frac{1}{\tau}\right)=e^{\pi i x^2/\tau-\pi i/4}\sqrt{\tau}\vartheta_2(x;\tau)$$
$$\vartheta_1\left(\frac{x}{\tau};-\frac{1}{\tau}\right)=-ie^{\pi i x^2/\tau-\pi i/4}\sqrt{\tau}\vartheta_1(x;\tau)$$
$$\vartheta_2\left(\frac{x}{\tau};-\frac{1}{\tau}\right)=e^{\pi i x^2/\tau-\pi i/4}\sqrt{\tau}\vartheta_0(x;\tau)$$
$$\vartheta_3\left(\frac{x}{\tau};-\frac{1}{\tau}\right)=e^{\pi i x^2/\tau-\pi i/4}\sqrt{\tau}\vartheta_3(x;\tau)$$

Theorem 7.5 (Landen の公式)**.**

$$\vartheta_0(0;2\tau)\vartheta_0(2x;2\tau)=\vartheta_0(x;\tau)\vartheta_4(x;\tau)$$
$$\vartheta_0(0;2\tau)\vartheta_1(2x;2\tau)=\vartheta_1(x;\tau)\vartheta_2(x;\tau)$$

Theorem 7.6.

$$\vartheta_0^2\vartheta_0(x)^2+\vartheta_2^2\vartheta_2(x)^2=\vartheta_3^2\vartheta_3(x)^2$$
$$\vartheta_0^2\vartheta_1(x)^2+\vartheta_3^2\vartheta_2(x)^2=\vartheta_2^2\vartheta_3(x)^2$$
$$\vartheta_0^2\vartheta_2(x)^2+\vartheta_3^2\vartheta_1(x)^2=\vartheta_2^2\vartheta_0(x)^2$$
$$\vartheta_0^2\vartheta_3(x)^2+\vartheta_2^2\vartheta_1(x)^2=\vartheta_3^2\vartheta_0(x)^2$$
$$\vartheta_0^4+\vartheta_2^4=\vartheta_3^4$$

7.2 Ramanujan テータ関数

Definition 7.7.

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}$$

Theorem 7.8.

$$f(a,0)=a+1$$
$$f(0,b)=b+1$$
$$f(a,b)=f(b,a)$$

Theorem 7.9.

$$f(a,b)=(-a,-b,ab;ab)_\infty$$

Theorem 7.10 (積分表示)**.**

$$\begin{aligned} f(a,b) = 1 + \frac{2a}{\sqrt{2\pi}} \int_0^\infty \frac{1 - a\sqrt{ab}\cosh(t\sqrt{\ln ab})}{1 + a^3 b - 2a\sqrt{ab}\cosh(t\sqrt{\ln ab})} e^{-t^2/2}\, dt \\ + \frac{2b}{\sqrt{2\pi}} \int_0^\infty \frac{1 - b\sqrt{ab}\cosh(t\sqrt{\ln ab})}{1 + ab^3 - 2b\sqrt{ab}\cosh(t\sqrt{\ln ab})} e^{-t^2/2}\, dt \end{aligned}$$

7.3 Dedekind イータ関数

Definition 7.11. $q = e^{\pi i \tau}$ として,

$$\begin{aligned} \eta(\tau) &:= q^{1/12} \prod_{0<n} (1 - q^{2n}) \\ \Delta(\tau) &:= (2\pi)^{12} \eta(\tau)^{24} \end{aligned}$$

と定義する.

Theorem 7.12 (特殊値)**.**

$$\begin{aligned} \eta(i) &= \frac{1}{2\pi^{3/4}} \Gamma\left(\frac{1}{4}\right) \\ \eta(2i) &= \frac{1}{2^{11/8}\pi^{3/4}} \Gamma\left(\frac{1}{4}\right) \\ \eta(3i) &= \frac{1}{2\sqrt[3]{3}\sqrt[12]{3 + 2\sqrt{3}}\pi^{3/4}} \Gamma\left(\frac{1}{4}\right) \\ \eta(4i) &= \frac{\sqrt[4]{\sqrt{2} - 1}}{2^{29/16}\pi^{3/4}} \Gamma\left(\frac{1}{4}\right) \\ \eta\left(\frac{i}{2}\right) &= \frac{1}{2^{7/8}\pi^{3/4}} \Gamma\left(\frac{1}{4}\right) \\ \eta(e^{2\pi i/3}) &= e^{-\pi i/24} \frac{\sqrt[8]{3}}{2\pi} \Gamma\left(\frac{1}{3}\right)^{3/2} \end{aligned}$$

Theorem 7.13 (関数等式)**.**

$$\begin{aligned} \eta(\tau + 1) &= e^{\pi i/12} \eta(\tau) \\ \eta\left(-\frac{1}{\tau}\right) &= e^{-\pi i/4} \sqrt{\tau}\, \eta(\tau) \end{aligned}$$

Theorem 7.14 (Fourier 級数表示)**.**

$$\Delta(\tau) = (2\pi)^{12} \sum_{0<n} \tau(n) e^{2\pi i n \tau}$$

7.4 楕円ラムダ関数

Definition 7.15.

$$\lambda(\tau) := \frac{\vartheta_2(0;\tau)^4}{\vartheta_3(0;\tau)^4}$$
$$\lambda^*(r) := \sqrt{\lambda(i\sqrt{r})}$$

Theorem 7.16 (特殊値)**.**

$$
\begin{aligned}
\lambda^*(1) &= \frac{1}{\sqrt{2}} \\
\lambda^*(2) &= \sqrt{2}-1 \\
\lambda^*(3) &= \frac{\sqrt{6}-\sqrt{2}}{4} \\
\lambda^*(4) &= 3-2\sqrt{2} \\
\lambda^*(5) &= \sqrt{\frac{1}{2}-\sqrt{\sqrt{5}-2}} \\
\lambda^*(6) &= (2-\sqrt{3})(\sqrt{3}-\sqrt{2}) \\
\lambda^*(7) &= \frac{1}{4\sqrt{2}}(3-\sqrt{7}) \\
\lambda^*(8) &= \left(1+\sqrt{2}-\sqrt{2+2\sqrt{2}}\right)^2 \\
\lambda^*(9) &= \frac{1}{2}(\sqrt{2}-\sqrt[4]{3})(\sqrt{3}-1) \\
\lambda^*(10) &= (\sqrt{10}-3)(\sqrt{2}-1)^2 \\
\lambda^*(12) &= (\sqrt{3}-\sqrt{2})^2(\sqrt{2}-1)^2 \\
\lambda^*\left(\frac{1}{2}\right) &= \sqrt{2(\sqrt{2}-1)} \\
\lambda^*\left(\frac{1}{3}\right) &= \frac{1}{2}\sqrt{2+\sqrt{3}} \\
\lambda^*\left(\frac{2}{3}\right) &= (2-\sqrt{3})(\sqrt{2}+\sqrt{3}) \\
\lambda^*\left(\frac{1}{4}\right) &= 2\sqrt{3\sqrt{2}-4} \\
\lambda^*\left(\frac{1}{5}\right) &= \sqrt{\frac{1}{2}+\sqrt{\sqrt{5}-2}} \\
\lambda^*\left(\frac{2}{5}\right) &= (\sqrt{10}-3)(\sqrt{2}+1)^2
\end{aligned}
$$

7.5　楕円アルファ関数

Definition 7.17. $k_r = \lambda^*(r)$ として，

$$\begin{aligned}
\alpha(r) &:= \frac{\pi}{4K(k_r)^2} + \sqrt{r}\left(1 - \frac{E(k_r)}{K(k_r)}\right) \\
\delta(r) &:= \sqrt{r} - 2\alpha(r)
\end{aligned}$$

と定義する．

Theorem 7.18 (特殊値)**.**

$$\begin{aligned}
\alpha(1) &= \frac{1}{2} \\
\alpha(2) &= \sqrt{2} - 1 \\
\alpha(3) &= \frac{\sqrt{3} - 1}{2} \\
\alpha(4) &= 2(\sqrt{2} - 1)^2 \\
\alpha(5) &= \frac{\sqrt{5} - \sqrt{2\sqrt{5} - 2}}{2} \\
\alpha(6) &= 5\sqrt{6} + 6\sqrt{3} - 8\sqrt{2} - 11 \\
\alpha(7) &= \frac{\sqrt{7}}{2} - 1 \\
\alpha(8) &= 2\left(10 + 7\sqrt{2}\right)\left(1 - \sqrt{2\sqrt{2} - 2}\right)^2 \\
\alpha(9) &= \frac{3 - 3^{3/4}\sqrt{2}(\sqrt{3} - 1)}{2} \\
\alpha(10) &= 72\sqrt{2} - 46\sqrt{5} + 33\sqrt{10} - 103 \\
\alpha(12) &= 264 - 188\sqrt{2} + 154\sqrt{3} - 108\sqrt{6} \\
\alpha(15) &= \frac{\sqrt{15} - \sqrt{5} - 1}{2}
\end{aligned}$$

Theorem 7.19.

$$\alpha(4r) = (1 + k_{4r})^2\alpha(r) - 2\sqrt{r}k_{4r}$$

7.6 楕円積分

Definition 7.20. 不完全楕円積分を以下で定義する.

$$
\begin{aligned}
F(\phi, k) &:= \int_0^{\phi} \frac{1}{\sqrt{1-k^2\sin^2 x}}\,dx \\
F(x; k) &:= \int_0^{x} \frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}}\,dz \\
E(\phi, k) &:= \int_0^{\phi} \sqrt{1-k^2\sin^2 x}\,dz \\
E(x; k) &:= \int_0^{x} \sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz
\end{aligned}
$$

Definition 7.21. 完全楕円積分を以下で定義する.

$$
\begin{aligned}
K(k) &:= F(1; k) \\
E(k) &:= E(1; k)
\end{aligned}
$$

Theorem 7.22 (特殊値)**.** Elliptic modulus を $k_r = \lambda^*(r)$ として,

$$K(k_1) = \frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2$$
$$K(k_2) = \frac{\sqrt{1+\sqrt{2}}}{8\sqrt[4]{2}\sqrt{\pi}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)$$
$$K(k_3) = \frac{\sqrt[4]{3}}{4\sqrt[3]{2}\pi}\Gamma\left(\frac{1}{3}\right)^3$$
$$K(k_4) = \sqrt{1+\sqrt{2}}8\sqrt{2\pi}\Gamma\left(\frac{1}{4}\right)^2$$
$$K(k_5) = \frac{\sqrt[4]{2+\sqrt{5}}}{4}\sqrt{\frac{\Gamma\left(\frac{1}{20}\right)\Gamma\left(\frac{3}{20}\right)\Gamma\left(\frac{7}{20}\right)\Gamma\left(\frac{9}{20}\right)}{10\pi}}$$
$$K(k_6) = \frac{\sqrt{(\sqrt{2}-1)(\sqrt{2}+\sqrt{3})(2+\sqrt{3})}}{8}\sqrt{\frac{\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right)}{6\pi}}$$
$$K(k_7) = \frac{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)}{4\pi\sqrt[4]{7}}$$
$$K(k_8) = \sqrt{\frac{1}{2}+\frac{\sqrt{1+5\sqrt{2}}}{4\sqrt{2}}}\frac{\sqrt[4]{1+\sqrt{2}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{8\sqrt{\pi}}$$
$$K(k_9) = \frac{\sqrt[4]{3}2+\sqrt{3}}{12\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2$$
$$K(k_{12}) = \frac{\sqrt[4]{3}(1+\sqrt{2})(\sqrt{2}+\sqrt{3})\sqrt{2-\sqrt{3}}}{2^{13/3}\pi}\Gamma\left(\frac{1}{3}\right)^3$$
$$K(k_{16}) = \frac{(1+\sqrt[4]{2})^2}{16\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^2$$
$$K(k_{25}) = \frac{2+\sqrt{5}}{20\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2$$

Theorem 7.23 (Legendre's relation)**.**

$$K(k)E\left(\sqrt{1-k^2}\right)+E(k)K\left(\sqrt{1-k^2}\right)-K(k)K\left(\sqrt{1-k^2}\right)=\frac{\pi}{2}$$

7.7 Jacobi の楕円関数

Definition 7.24.

$$x = F(\phi, k)$$

であるとき, その逆関数として,

$$\phi := \mathrm{am}(x, k)$$

が定まる.

$$\begin{aligned}
\mathrm{sn}(x, k) &:= \sin(\mathrm{am}(x, k)) \\
\mathrm{cn}(x, k) &:= \cos(\mathrm{am}(x, k)) \\
\mathrm{dn}(x, k) &:= \sqrt{1 - k^2 \,\mathrm{sn}^2(x, k)}
\end{aligned}$$

Theorem 7.25 (k に関する特殊値)**.**

$$\begin{aligned}
\mathrm{sn}(x, 0) &= \sin x \\
\mathrm{cn}(x, 0) &= \cos x \\
\mathrm{dn}(x, 0) &= 1 \\
\mathrm{sn}(x, 1) &= \tanh x \\
\mathrm{cn}(x, 1) &= \frac{1}{\cosh x} \\
\mathrm{dn}(x, 1) &= \frac{1}{\cosh x}
\end{aligned}$$

Theorem 7.26 (x に関する特殊値)**.** $K := K(k)$, sn $x := \mathrm{sn}(x, k)$ とする.

$$\begin{aligned}
\mathrm{sn}\, 0 &= 0 \\
\mathrm{cn}\, 0 &= 1 \\
\mathrm{dn}\, 0 &= 1 \\
\mathrm{sn}\, K &= 1 \\
\mathrm{cn}\, K &= 0 \\
\mathrm{dn}\, K &= k' \\
\mathrm{sn}\, \frac{K}{2} &= \frac{1}{\sqrt{1 + k'}} \\
\mathrm{cn}\, \frac{K}{2} &= \sqrt{\frac{k'}{1 + k'}} \\
\mathrm{dn}\, \frac{K}{2} &= \sqrt{k'}
\end{aligned}$$

Theorem 7.27 (加法定理)**.**

$$\begin{aligned}
\operatorname{sn}(x+y) &= \frac{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y} \\
\operatorname{cn}(x+y) &= \frac{\operatorname{cn} x \operatorname{cn} y - \operatorname{sn} x \operatorname{sn} y \operatorname{dn} x \operatorname{dn} y}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y} \\
\operatorname{dn}(x+y) &= \frac{\operatorname{dn} x \operatorname{dn} y - k^2 \operatorname{sn} x \operatorname{sn} y \operatorname{cn} x \operatorname{cn} y}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}
\end{aligned}$$

Theorem 7.28 (倍角公式)**.**

$$\begin{aligned}
\operatorname{sn} 2x &= \frac{2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^4 x} \\
\operatorname{cn} 2x &= \frac{1 - 2 \operatorname{sn}^2 x + k^2 \operatorname{sn}^4 x}{1 - k^2 \operatorname{sn}^4 x} \\
\operatorname{dn} 2x &= \frac{1 - 2k^2 \operatorname{sn}^2 x + k^2 \operatorname{sn}^4 x}{1 - k^2 \operatorname{sn}^4 x}
\end{aligned}$$

Theorem 7.29.

$$\begin{aligned}
\operatorname{sn}^2 x &= \frac{1 - \operatorname{cn} 2x}{1 + \operatorname{dn} 2x} \\
\operatorname{cn}^2 x &= \frac{\operatorname{dn} 2x + \operatorname{cn} 2x}{1 + \operatorname{dn} 2x} \\
\operatorname{dn}^2 x &= \frac{\operatorname{dn} 2x + \operatorname{cn} 2x}{1 + \operatorname{cn} 2x}
\end{aligned}$$

Theorem 7.30 (Jacobi の虚数変換式)**.**

$$\begin{aligned}
\operatorname{sn}(ix, k) &= i \frac{\operatorname{sn}(x, k')}{\operatorname{cn}(x, k')} \\
\operatorname{cn}(ix, k) &= \frac{1}{\operatorname{cn}(x, k')} \\
\operatorname{dn}(ix, k) &= \frac{\operatorname{dn}(x, k')}{\operatorname{cn}(x, k')}
\end{aligned}$$

Definition 7.31 (Jacobi の第 2 種楕円関数)**.**

$$\begin{aligned}
\boldsymbol{\varepsilon}(x, k) &:= \int_0^x \operatorname{dn}(t, k)^2 \, dt \\
\boldsymbol{Z}(x, k) &:= \boldsymbol{\varepsilon}(x, k) - \frac{E(k)}{K(k)} x
\end{aligned}$$

Theorem 7.32 (加法定理)**.**

$$\begin{aligned}
\boldsymbol{\varepsilon}(x+y) &= \boldsymbol{\varepsilon}(x) + \boldsymbol{\varepsilon}(y) - k \operatorname{sn} x \operatorname{sn} y \operatorname{sn}(x+y) \\
\boldsymbol{Z}(x+y) &= \boldsymbol{Z}(x) + \boldsymbol{Z}(y) - k \operatorname{sn} x \operatorname{sn} y \operatorname{sn}(x+y)
\end{aligned}$$

7.8 Weierstrass の楕円関数

Definition 7.33.

$$\wp(x;\omega_1,\omega_2) := \frac{1}{x^2} + \sum_{\substack{n,m\in\mathbb{Z}\\(n,m)\neq(0,0)}} \left(\frac{1}{(x+n\omega_1+m\omega_2)^2} - \frac{1}{(n\omega_1+m\omega_2)^2}\right)$$
$$\wp(x;\tau) := \wp(x;1,\tau)$$

Theorem 7.34 (微分方程式)**.**

$$\left(\frac{dy}{dx}\right)^2 = 4y^3 - g_2y - g_3$$

Theorem 7.35.

$$\wp(x;\omega_1,\omega_2) = \frac{1}{x^2} + \frac{g_2}{20}x^2 + \frac{g_3}{28}x^4 + O(x^6)$$

Definition 7.36. 方程式 $4x^3 - g_2x - g_3$ の根を e_1, e_2, e_3 とする.

Theorem 7.37.

$$e_1 + e_2 + e_3 = 0$$
$$2(e_1^2 + e_2^2 + e_3^2) = g_2$$
$$4e_1e_2e_3 = g_3$$

Theorem 7.38. $\omega_3 := -\omega_1 - \omega_2$ とする.

$$\wp\left(\frac{\omega_1}{2}\right) = e_1, \quad \wp\left(\frac{\omega_2}{2}\right) = e_2, \quad \wp\left(\frac{\omega_3}{2}\right) = e_3$$

7.9 Eisenstein 級数

Definition 7.39.

$$G_r(\tau) := \sum_{\substack{n,m\in\mathbb{Z}\\(n,m)\neq(0,0)}} \frac{1}{(n+m\tau)^r}$$
$$E_{2r} := \frac{G_{2r}(\tau)}{2\zeta(2r)}$$

Theorem 7.40. $q = e^{\pi i \tau}$ とする.

$$
\begin{aligned}
E_{2r}(q) &= 1 - \frac{4r}{B_{2r}} \sum_{0<n} \frac{n^{2r-1} q^{2n}}{1 - q^{2n}} \\
&= 1 - \frac{4r}{B_{2r}} \sum_{0<n} \sigma_{2r-1}(n) q^{2n}
\end{aligned}
$$

7.10 j 不変量

Definition 7.41.

$$
\begin{aligned}
g_2(\tau) &:= 60 G_4(\tau) \\
g_3(\tau) &:= 140 G_6(\tau)
\end{aligned}
$$

として,

$$
j(\tau) := \frac{1728 g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2}
$$

と定義する.

Theorem 7.42 (特殊値)**.**

$$
J(\tau) := \frac{j(\tau)}{1728}
$$

とする.

$$
\begin{aligned}
&J\left(\frac{1+i\sqrt{3}}{2}\right) = 0 \\
&J(i) = 1 \\
&J\left(\frac{1+i}{2}\right) = 1 \\
&J(\sqrt{2}i) = \left(\frac{5}{3}\right)^3 \\
&J(2i) = \left(\frac{11}{2}\right)^3 \\
&J(2\sqrt{2}i) = \left(\frac{5}{6}\right)^3 (19+13\sqrt{2})^3 \\
&J\left(\frac{1+2i\sqrt{2}}{3}\right) = \left(\frac{5}{6}\right)^3 (19-13\sqrt{2})^3 \\
&J(4i) = \frac{1}{4^3}(724+513\sqrt{2})^3 \\
&J\left(\frac{1}{2}+i\right) = \frac{1}{4^3}(724-513\sqrt{2})^3
\end{aligned}
$$

Theorem 7.43.

$$
\begin{aligned}
j(\tau+1) &= j(\tau) \\
j\left(-\frac{1}{\tau}\right) &= j(\tau)
\end{aligned}
$$

8 Bessel 関数系

8.1 Bessel 関数

Definition 8.1.

$$
\begin{aligned}
J_\alpha(x) &:= \frac{1}{\Gamma(1+\alpha)}\left(\frac{z}{2}\right)^\alpha {}_0F_1\left[\begin{matrix} - \\ 1+\alpha \end{matrix}; -\frac{x^2}{4}\right] \\
Y_\alpha(x) &:= \frac{J_\alpha(x)\cos\pi\alpha - J_{-\alpha}(x)}{\sin\pi\alpha}
\end{aligned}
$$

Theorem 8.2 (特殊値)**.**

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$
$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Theorem 8.3.

$$J_{-n}(x) = (-1)^n J_n(x)$$
$$Y_{-n}(x) = (-1)^n Y_n(x)$$

Theorem 8.4 (微分方程式)**.**

$$\left(x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (x^2 - \alpha^2) \right) y = 0$$

Theorem 8.5 (母関数)**.**

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Theorem 8.6 (級数表示)**.**

$$J_\alpha(x) = \sum_{0 \leq n} \frac{(-1)^n}{n! \Gamma(1 + \alpha + n)} \left(\frac{x}{2} \right)^{2n+\alpha}$$

Theorem 8.7 (積分表示)**.**

$$J_\alpha(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin t - \alpha t) \, dt$$

Theorem 8.8.

$$\sum_{n=-\infty}^{\infty} J_n(x) = 1$$
$$\sum_{n=-\infty}^{\infty} J_n(x)^2 = 1$$

Theorem 8.9 (加法定理)**.**

$$J_n(x + y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y)$$

8.2 Hankel 関数

Definition 8.10.

$$H_\alpha^{(1)}(x) := J_\alpha(x) + iY_\alpha(x)$$
$$H_\alpha^{(2)}(x) := J_\alpha(x) - iY_\alpha(x)$$

Theorem 8.11.

$$H_\alpha^{(1)}(x) = -\frac{2i}{\sqrt{\pi}}(2x)^\alpha e^{-i\pi\alpha+ix} U\left(\frac{1}{2}+\alpha, 1+2\alpha, -2iz\right)$$
$$H_\alpha^{(2)}(x) = \frac{2i}{\sqrt{\pi}}(2x)^\alpha e^{i\pi\alpha-ix} U\left(\frac{1}{2}+\alpha, 1+2\alpha, 2iz\right)$$

8.3 変形 Bessel 関数

Definition 8.12.

$$J_\alpha(x) := \frac{1}{\Gamma(1+\alpha)}\left(\frac{z}{2}\right)^\alpha {}_0F_1\left[\begin{matrix} - \\ 1+\alpha \end{matrix}; \frac{x^2}{4}\right]$$
$$Y_\alpha(x) := \frac{\pi}{2}\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin\pi\alpha}$$

Theorem 8.13.

$$I_{-n}(x) = I_n(x)$$
$$K_{-\alpha}(x) = K_\alpha(x)$$

Theorem 8.14 (微分方程式)**.**

$$\left(x^2\frac{d^2}{dx^2} + x\frac{d}{dx} - (x^2+\alpha^2)\right)y = 0$$

8.4 球 Bessel 関数

Definition 8.15.

$$
\begin{aligned}
j_\alpha(x) &:= \sqrt{\frac{\pi}{2x}} J_{1/2+\alpha}(x) \\
y_\alpha(x) &:= \sqrt{\frac{\pi}{2x}} Y_{1/2+\alpha}(x)
\end{aligned}
$$

Theorem 8.16 (微分方程式)**.**

$$
\left(x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} + (x^2 - \alpha(1+\alpha)) \right) y = 0
$$

Definition 8.17 (球 Hankel 関数)**.**

$$
\begin{aligned}
h_\alpha^{(1)}(x) &:= j_\alpha(x) + i y_\alpha(x) \\
h_\alpha^{(2)}(x) &:= j_\alpha(x) - i y_\alpha(x)
\end{aligned}
$$

8.5 変形球 Bessel 関数

Definition 8.18.

$$
\begin{aligned}
i_\alpha(x) &:= \sqrt{\frac{\pi}{2x}} I_{1/2+\alpha}(x) \\
k_\alpha(x) &:= \sqrt{\frac{2}{\pi x}} K_{1/2+\alpha}(x)
\end{aligned}
$$

8.6 Airy 関数

Definition 8.19.

$$
\begin{aligned}
\mathrm{Ai}(x) &:= \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} {}_0F_1\left[\begin{matrix} - \\ \frac{2}{3} \end{matrix}; \frac{x^3}{9}\right] - \frac{x}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} {}_0F_1\left[\begin{matrix} - \\ \frac{4}{3} \end{matrix}; \frac{x^3}{9}\right] \\
\mathrm{Bi}(x) &:= \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)} {}_0F_1\left[\begin{matrix} - \\ \frac{2}{3} \end{matrix}; \frac{x^3}{9}\right] + \frac{3^{1/6}x^3}{\Gamma\left(\frac{1}{3}\right)} {}_0F_1\left[\begin{matrix} - \\ \frac{4}{3} \end{matrix}; \frac{x^3}{9}\right]
\end{aligned}
$$

Definition 8.20 (Scorer 関数)**.**

$$\mathrm{Gi}(x) := \mathrm{Bi}(x) - \mathrm{Hi}(x)$$
$$\mathrm{Hi}(x) := \frac{2}{3}\mathrm{Bi}(x) + \frac{x^2}{2\pi}\,{}_1F_2\left[\begin{matrix}1\\ \frac{4}{3},\frac{5}{3}\end{matrix};\frac{x^3}{9}\right]$$

Theorem 8.21 (特殊値)**.**

$$\mathrm{Ai}(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}$$
$$\mathrm{Bi}(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)}$$

Theorem 8.22 (微分方程式)**.**

$$\left(\frac{d^2}{dx^2} - x\right) y = 0$$

Theorem 8.23.

$$\mathrm{Ai}(x) = \frac{\sqrt{x}}{3}\left(I_{-1/3}\left(\frac{2}{3}x^{3/2}\right) - I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right)$$
$$\mathrm{Bi}(x) = \sqrt{\frac{x}{3}}\left(I_{-1/3}\left(\frac{2}{3}x^{3/2}\right) + I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right)$$

Theorem 8.24 (Airy 積分)**.**

$$\int_0^\infty \cos(t^3 + xt)\,dt = \frac{\pi}{\sqrt[3]{3}}\mathrm{Ai}\left(\frac{x}{\sqrt[3]{3}}\right)$$
$$\int_0^\infty \sin(t^3 + xt)\,dt = \frac{\pi}{\sqrt[3]{3}}\mathrm{Gi}\left(\frac{x}{\sqrt[3]{3}}\right)$$
$$\int_0^\infty e^{-t^3+xt}\,dt = \frac{\pi}{\sqrt[3]{3}}\mathrm{Hi}\left(\frac{x}{\sqrt[3]{3}}\right)$$

9 積分関数系

9.1 誤差関数

Definition 9.1.

$$\operatorname{erf} x := \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz$$
$$\operatorname{erfc} x := 1 - \operatorname{erf} x$$

Theorem 9.2 (特殊値).

$$\operatorname{erf} 0 = 0$$
$$\operatorname{erf} \infty = 1$$

Theorem 9.3 (級数表示).

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \sum_{0 \le n} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$
$$\operatorname{erf} x = \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{0 \le n} \frac{(2x)^{2n+1}}{(2n+1)!!}$$

Theorem 9.4 ($x = \infty$ における漸近展開).

$$\operatorname{erf} x \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{0 \le n} \frac{(-1)^n (2n-1)!!}{2^n} x^{-2n-1}$$

9.2 指数積分

Definition 9.5.

$$E_n(x) := \int_1^\infty \frac{e^{-xt}}{t^n} \, dt$$
$$\operatorname{Ei}(x) := -E_1(x)$$
$$\operatorname{Ein}(x) := \int_0^x \frac{1 - e^{-t}}{t} \, dt$$

Theorem 9.6 (特殊値).

$$E_n(0) = \frac{1}{n-1}$$

Theorem 9.7.

$$\operatorname{Ein}(x) = \gamma + \ln x + E_1(x)$$

Theorem 9.8 (級数表示).

$$E_1(x) = -\gamma - \ln x + \sum_{0 < n} \frac{(-1)^{n-1}}{n!n} x^n$$

Theorem 9.9 (微分).

$$E_n'(x) = -E_{n-1}(x)$$

9.3　正弦積分

Definition 9.10.

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\,dt$$
$$\mathrm{Shi}(x) := \int_0^x \frac{\sinh t}{t}\,dt$$

Theorem 9.11 (級数表示)**.**

$$\mathrm{Si}(x) = \sum_{0\le n} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1}$$
$$\mathrm{Shi}(x) = \sum_{0\le n} \frac{x^{2n+1}}{(2n+1)(2n+1)!}$$

Theorem 9.12 (微分)**.**

$$\frac{d}{dx}\,\mathrm{Si}(x) = \frac{\sin x}{x}$$
$$\frac{d}{dx}\,\mathrm{Shi}(x) = \frac{\sinh x}{x}$$

9.4　余弦積分

Definition 9.13.

$$\mathrm{Ci}(x) := -\int_x^\infty \frac{\cos t}{t}\,dt$$
$$\mathrm{Cin}(x) := \int_0^x \frac{1-\cos t}{t}\,dt$$
$$\mathrm{Chi}(x) := \gamma + \ln x + \int_0^x \frac{\cosh t - 1}{t}\,dt$$

Theorem 9.14.

$$\mathrm{Ci}(x) = \gamma + \ln x - \mathrm{Cin}(x)$$
$$\mathrm{Ci}(x) = \frac{\mathrm{Ei}(ix) + \mathrm{Ei}(-ix)}{2}$$

Theorem 9.15 (級数表示)**.**

$$\operatorname{Ci}(x) = \gamma + \ln x + \sum_{0<n} \frac{(-1)^n}{2n(2n)!} x^{2n}$$

Theorem 9.16 (微分)**.**

$$\frac{d}{dx}\operatorname{Ci}(x) = \frac{\cos x}{x}$$
$$\frac{d}{dx}\operatorname{Chi}(x) = \frac{\cosh x}{x}$$

9.5 対数積分

Definition 9.17. Soldner's constant を μ として,

$$\operatorname{li} x := \int_{\mu}^{x} \frac{1}{\ln t}\, dt$$

と定義する.

Theorem 9.18 (特殊値)**.**

$$\operatorname{li} 0 = 0$$
$$\operatorname{li} 1 = -\infty$$
$$\operatorname{li} \mu = 0$$

Theorem 9.19 (微分)**.**

$$\frac{d}{dx}\operatorname{li} x = \frac{1}{\ln x}$$

Theorem 9.20.

$$\operatorname{li} x = \operatorname{Ei}(\ln x)$$

Theorem 9.21 (級数表示)**.**

$$\operatorname{li} x = \gamma + \ln\ln x + \sum_{0<n} \frac{\ln^n x}{n!n}$$
$$\operatorname{li} x = \gamma + \ln\ln x + \sqrt{x} \sum_{0<n} \frac{(-1)^{n-1}\ln^n x}{2^{n-1}n} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}$$

9.6　不完全ガンマ関数

Definition 9.22.

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-x}\,dt$$
$$\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-x}\,dt$$

Theorem 9.23 (特殊値)**.**

$$\Gamma(0,x) = E_1(x)$$
$$\Gamma(1,x) = e^{-x}$$
$$\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi}\,\operatorname{erfc} x$$
$$\gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi}\,\operatorname{erf} x$$

Theorem 9.24 (微分方程式)**.**

$$\left(\frac{d^2}{dx^2} + \left(1 + \frac{1-s}{x}\right)\frac{d}{dx}\right) y = 0$$

Theorem 9.25.

$$\gamma(s,x) + \Gamma(s,x) = \Gamma(s)$$

Theorem 9.26.

$$\gamma(s,x) = \frac{x^s e^{-x}}{s} M(1, 1+s, x)$$
$$\Gamma(s,x) = x^s e^{-x} U(1, 1+s, x)$$

9.7　不完全ベータ関数

Definition 9.27.

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

Theorem 9.28 (特殊値)**.**

$$\begin{aligned} B_x(a,1) &= \frac{x^a}{a} \\ B_x(1,b) &= \frac{1-(1-x)^b}{b} \\ B_x\left(\frac{1}{2},0\right) &= 2\operatorname{artanh}\sqrt{x} \\ B_x\left(\frac{1}{2},\frac{1}{2}\right) &= 2\arcsin\sqrt{x} \end{aligned}$$

Theorem 9.29 (関数等式)**.**

$$\begin{aligned} B_x(a,b) + B_{1-x}(b,a) &= B(a,b) \\ B_x(a+1,b) &= \frac{1}{a+b}\left(aB_x(a,b) - x^a(1-x)^b\right) \\ B_x(a,b+1) &= \frac{1}{a+b}\left(bB_x(a,b) + x^a(1-x)^b\right) \end{aligned}$$

Theorem 9.30.

$$B_x(a,b) = \frac{1}{a}x^a(1-x)^b {}_2F_1\left[\begin{matrix}1, a+b \\ 1+a\end{matrix}; x\right]$$

10 多項式系

10.1 Bernoulli 多項式

Definition 10.1.

$$\begin{aligned} \frac{te^{xt}}{e^t-1} &= \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \\ B_n &:= B_n(0) \end{aligned}$$

Theorem 10.2.

$$
\begin{aligned}
B_0(0) &= 1 \\
B_1(x) &= x - \frac{1}{2} \\
B_2(x) &= x^2 - x + \frac{1}{6} \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30} \\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \\
B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}
\end{aligned}
$$

Theorem 10.3 (明示公式)**.**

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k$$

Theorem 10.4 (Faulhaber's formula)**.**

$$\sum_{k=0}^{n} k^r = \frac{1}{r+1}(B_{r+1}(n+1) - B_r)$$

Theorem 10.5.

$$
\begin{aligned}
B_n(1-x) &= (-1)^n B_n(x) \\
(-1)^n B_n(x) &= B_n(x) + nx^{n-1}
\end{aligned}
$$

Theorem 10.6 (Fourier 級数)**.**

$$B_r(x) = -\frac{2r!}{(2\pi)^r} \sum_{0<n} \frac{1}{n^r} \cos\left(2\pi nx - \frac{\pi r}{2}\right)$$

Theorem 10.7 (乗法定理)**.**

$$B_r(nx) = n^{r-1} \sum_{k=0}^{n-1} B_r\left(x + \frac{k}{n}\right)$$

10.2 Chebyshev 多項式

Definition 10.8.

$$T_n(\cos x) := \cos nx$$
$$U_n(\cos x) := \frac{\sin(n+1)x}{\sin x}$$

Theorem 10.9.

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

Theorem 10.10.

$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_2(x) = 4x^2 - 1$$
$$U_3(x) = 8x^3 - 4x$$
$$U_4(x) = 16x^4 - 12x^2 + 1$$
$$U_5(x) = 32x^5 - 32x^3 + 6x$$
$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$$

Theorem 10.11 (母関数)**.**

$$\frac{1-xt}{1-2xt+t^2} = \sum_{0\le n} T_n(x)t^n$$
$$\frac{1-t^2}{1-2xt+t^2} = T_0(x) + 2\sum_{0<n} T_n(x)t^n$$
$$\frac{1}{1-2xt+t^2} = \sum_{0\le n} U_n(x)t^n$$

Theorem 10.12.

$$T_n(x) = \frac{x^2}{2}\left(\left(1+\sqrt{1-\frac{1}{x^2}}\right)^n + \left(\sqrt{1-\frac{1}{x^2}}\right)^n\right)$$
$$T_n(x) = \frac{(-1)^n\sqrt{\pi(1-x^2)}}{2^n\Gamma\left(\frac{1}{2}+n\right)}\frac{d^n}{dx^n}(1-x^2)^{n-1/2}$$
$$T_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}x^{n-2k}(x^2-1)^k$$
$$T_n(x) = 2^{n-1}\prod_{k=0}^{n-1}\left(x-\cos\frac{(2k+1)\pi}{2n}\right)$$
$$U_n(x) = \frac{(-1)^n(n+1)\sqrt{\pi}}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)\sqrt{1-x^2}}\frac{d^n}{dx^n}(1-x^2)^{n+1/2}$$
$$U_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}(2x)^{n-2k}$$
$$U_n(x) = 2^n\prod_{k=1}^{n}\left(x-\cos\frac{\pi k}{n+1}\right)$$

Theorem 10.13.

$$T_n(x) = {}_2F_1\left[\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}; \frac{1-x}{2}\right]$$
$$U_n(x) = (n+1)\,{}_2F_1\left[\begin{matrix}-n, n+2\\ \frac{3}{2}\end{matrix}; \frac{1-x}{2}\right]$$

Theorem 10.14 (直交性)**.**

$$\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases}\frac{\pi}{2}\delta_{n,m}, & n\neq 0,\, m\neq 0\\ \pi, & n=m=0\end{cases}$$

Theorem 10.15 (漸化式)**.**

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$
$$T_{n+1}(x) = xT_n(x) - \sqrt{(1-x^2)(1-T_n(x)^2)}$$
$$(x-1)(T_{2n+1}(x)-1) = (T_{n+1}(x)-T_n(x))^2$$
$$2(x^2-1)(T_{2n}(x)-1) = (T_{n+1}(x)-T_{n-1}(x))^2$$

10.3 Legendre 多項式

Definition 10.16.

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Theorem 10.17 (母関数)**.**

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{0 \le n} P_n(x) t^n$$
$$e^{xt} J_0 \left(t\sqrt{1 - x^2} \right) = \sum_{0 \le n} \frac{P_n(x)}{n!} t^n$$

Theorem 10.18.

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n - 2k}{n} x^{n-2k}$$
$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x - 1)^k (x + 1)^{n-k}$$

Theorem 10.19.

$$P_n(x) = \left(\frac{x - 1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -n \\ 1 \end{matrix} ; -\frac{1 + x}{1 - x} \right]$$
$$P_n(x) = \binom{2n}{n} \left(\frac{x}{2} \right)^n {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ \frac{1}{2} - n \end{matrix} ; \frac{1}{x^2} \right]$$
$$P_n(x) = {}_2F_1 \left[\begin{matrix} -n, n + 1 \\ 1 \end{matrix} ; \frac{1 - x}{2} \right]$$

Theorem 10.20 (漸化式)**.**

$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0$$
$$(1 - x^2)P_n'(x) + nxP_n(x) - nP_{n-1}(x) = 0$$

10.4 Hermite 多項式

Definition 10.21.

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Theorem 10.22.

$$\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= 2x\\
H_2(x) &= 4x^2 - 2\\
H_3(x) &= 8x^3 - 12x\\
H_4(x) &= 16x^4 - 48x^2 + 12\\
H_5(x) &= 32x^5 - 160x^3 + 120x\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120\\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x
\end{aligned}$$

Theorem 10.23 (母関数)**.**

$$e^{2xt-t^2} = \sum_{0\le n} \frac{H_n(x)}{n!} t^n$$

Theorem 10.24 (直交性)**.**

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = 2^n n!\sqrt{\pi}\delta_{n,m}$$

Theorem 10.25 (漸化式)**.**

$$\begin{aligned}
&H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x)\\
&H_n'(x) = 2nH_{n-1}(x)
\end{aligned}$$

Theorem 10.26 (加法定理)**.**

$$\begin{aligned}
2^{n/2}H_n\left(\frac{x+y}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}H_k(x)H_{n-k}(y)\\
H_n(x+y) &= \sum_{k=0}^{n}\binom{n}{k}H_k(x)(2y)^{n-k}
\end{aligned}$$

Theorem 10.27.

$$\begin{aligned}
H_{2n}(x) &= (-1)^n 2^n (2n-1)!!\,{}_1F_1\left[\begin{matrix}-n\\ \frac{1}{2}\end{matrix}; x^2\right]\\
H_{2n+1}(x) &= (-1)^n 2^{n+1}(2n+1)!!x\,{}_1F_1\left[\begin{matrix}-n\\ \frac{3}{2}\end{matrix}; x^2\right]
\end{aligned}$$

10.5 Laguerre 多項式

Definition 10.28.

$$L_n(x) := \frac{e^x}{n!}\frac{d^n}{dx^n}x^n e^{-x}$$

Theorem 10.29.

$$\begin{aligned}
L_0(x) &= 1\\
L_1(x) &= 1 - x\\
L_2(x) &= 1 - 2x + \frac{1}{2}x^2\\
L_3(x) &= 1 - 3x + \frac{3}{2}x^2 - \frac{x^3}{6}
\end{aligned}$$

Theorem 10.30.

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n}{k}x^k$$

Theorem 10.31 (漸化式)**.**

$$\begin{aligned}
&(n+1)L_{n+1}(x) - (2n - x + 1)L_n(x) + nL_{n-1}(x) = 0\\
&xL_n'(x) - nL_n(x) + nL_{n-1}(x)
\end{aligned}$$

10.6 Jacobi 多項式

Definition 10.32.

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right)$$

Theorem 10.33.

$$\begin{aligned}
&P_0^{(\alpha,\beta)}(x) = 1\\
&P_1^{(\alpha,\beta)}(x) = 1 + \alpha - \frac{1}{2}\left(2 + \alpha + \beta\right)\left(1 - x\right)\\
&P_2^{(\alpha,\beta)}(x) = \frac{2+\alpha}{2}\left(1 + \alpha - (3 + \alpha + \beta)(1 - x)\right) + \frac{1}{8}(3 + \alpha + \beta)(4 + \alpha + \beta)(1 - x)^2
\end{aligned}$$

Theorem 10.34.

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\left[\begin{matrix}-n, 1+n+\alpha+\beta \\ 1+\alpha\end{matrix}; \frac{1-x}{2}\right]$$
$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, -n-\beta \\ 1+\alpha\end{matrix}; -\frac{1-x}{1+x}\right]$$

10.7 Gegenbauer 多項式

Definition 10.35.

$$C_n^{(\alpha)}(x) := \frac{(2\alpha)_n}{\left(\frac{1}{2}+\alpha\right)_n} P_n^{(\alpha-1/2,\alpha-1/2)}(x)$$

Theorem 10.36.

$$C_0^{(\alpha)}(x) = 1$$
$$C_1^{(\alpha)}(x) = 2\alpha x$$
$$C_2^{(\alpha)}(x) = 2\alpha(1+\alpha)x^2 - \alpha$$
$$C_3^{(\alpha)}(x) = \frac{4}{3}\alpha(1+\alpha)(2+\alpha)x^3 - 2\alpha(1+\alpha)x$$

Theorem 10.37 (母関数)**.**

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{0\le n} C_n^{(\alpha)}(x)t^n$$

Theorem 10.38 (漸化式)**.**

$$nC_n^{(\alpha)}(x) - 2(n+\alpha-1)xC_{n-1}^{(\alpha)}(x) + (n+2\alpha-2)C_{n-2}^{(\alpha)}(x) = 0$$

Theorem 10.39.

$$C_n^{(\alpha)}(x) = \binom{n+2\alpha-1}{n} {}_2F_1\left[\begin{matrix}-n, n+2\alpha \\ \frac{1}{2}+\alpha\end{matrix}; \frac{1-x}{2}\right]$$
$$C_n^{(\alpha)}(x) = 2^n\binom{n+\alpha-1}{n}(x-1)^n {}_2F_1\left[\begin{matrix}-n, \frac{1}{2}-n-\alpha \\ 1-2n-2\alpha\end{matrix}; \frac{2}{1-x}\right]$$
$$C_n^{(\alpha)}(x) = \binom{n+2\alpha-1}{n}\left(\frac{1+x}{2}\right)^n {}_2F_1\left[\begin{matrix}-n, \frac{1}{2}-n-\alpha \\ \frac{1}{2}+\alpha\end{matrix}; -\frac{1-x}{1+x}\right]$$

11 超幾何関数系

11.1 Pochhammer 記号

Definition 11.1.

$$(a)_0 = 1, \quad (a)_n = \prod_{k=0}^{n-1}(a+k), \quad (a)_{-n} = \frac{1}{\prod_{k=1}^{n}(a-k)}$$

と定義する. また,

$$(a_1, \ldots, a_r)_n = (a_1)_n \cdots (a_r)_n$$

と略記する.

Theorem 11.2.

$$\begin{aligned}
(a)_n &= (-1)^n(1-n-a)_n \\
(a)_{-n} &= \frac{(-1)^n}{(1-a)_n} \\
(a)_{n-k} &= \frac{(-1)^k(a)_n}{(1-n-a)_k} \\
(a)_{n+k} &= (a)_n(a+n)_k \\
(a)_{2n} &= 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{1+a}{2}\right)_n \\
\frac{(a+1)_n}{(a)_n} &= \frac{a+n}{a} \\
(-n)_k &= \frac{(-1)^k n!}{(n-k)!}
\end{aligned}$$

Theorem 11.3.

$$\sum_{k=1}^{r} a_k = \sum_{k=1}^{r} b_k$$

ならば、

$$\lim_{n\to\infty}\frac{(a_1,\ldots,a_r)_n}{(b_1,\ldots,b_r)_n} = \prod_{k=1}^{r}\frac{\Gamma(b_k)}{\Gamma(a_k)}$$

11.2 合流型超幾何関数

Definition 11.4.

$$M(a,b,x) := {}_1F_1\left[\begin{matrix} a \\ b \end{matrix}; x\right]$$
$$U(a,b,x) := \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a,b;x) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(1+a-b, 2-b, x)$$

Theorem 11.5 (特殊値)**.**

$$M(0,b,x) = 1$$
$$U(0,c,x) = 1$$
$$M(a,a,x) = e^x$$

Theorem 11.6 (微分方程式)**.**

$$\left(x\frac{d^2}{dx^2} + (b-x)\frac{d}{dx} - a \right) y = 0$$

Theorem 11.7 (積分表示)**.**

$$M(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} e^{xt}\, dt$$
$$U(a,b,x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1+t)^{b-a-1} e^{-xt}\, dt$$

Theorem 11.8.

$$M(a,2a,x) = \left(\frac{x}{4}\right)^{1/2-a} e^{x/2} \Gamma\left(\frac{1}{2}+a\right) I_{a-1/2}\left(\frac{x}{2}\right)$$
$$U(a,2a,x) = \frac{x^{1/2-a} e^{x/2}}{\sqrt{\pi}} K_{a-1/2}\left(\frac{x}{2}\right)$$

11.3 Whittaker 関数

Definition 11.9.

$$M_{a,b}(x) := x^{1/2+b} e^{-x/2} M\left(\frac{1}{2}-a+b, 1+2b, x\right)$$
$$W_{a,b}(x) := x^{1/2+b} e^{-x/2} U\left(\frac{1}{2}-a+b, 1+2b, x\right)$$

Theorem 11.10 (微分方程式)**.**

$$\left(\frac{d^2}{dx^2}-\frac{1}{4}+\frac{a}{x}+\frac{1-4b^2}{4x^2}\right)y=0$$

11.4 Gauss の超幾何関数

Definition 11.11.

$${}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\b_1,\dots,b_s\end{matrix};x\right]:=\sum_{0\le n}\frac{(a_1,\dots,a_r)_n}{(b_1,\dots,b_s)_n n!}x^n$$

Theorem 11.12.

$$\begin{aligned}
e^x&={}_0F_0\left[\begin{matrix}-\\-\end{matrix};x\right]\\
\sinh x&=x\,{}_0F_1\left[\begin{matrix}-\\\frac{3}{2}\end{matrix};\frac{x^2}{4}\right]\\
\cosh x&={}_0F_1\left[\begin{matrix}-\\\frac{1}{2}\end{matrix};\frac{x^2}{4}\right]\\
\sin x&=x\,{}_0F_1\left[\begin{matrix}-\\\frac{3}{2}\end{matrix};-\frac{x^2}{4}\right]\\
\cos x&={}_0F_1\left[\begin{matrix}-\\\frac{1}{2}\end{matrix};-\frac{x^2}{4}\right]\\
\ln(1+x)&=x\,{}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};-x\right]\\
\arcsin x&=x\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\\frac{3}{2}\end{matrix};x^2\right]\\
\arctan x&=x\,{}_2F_1\left[\begin{matrix}1,\frac{1}{2}\\\frac{3}{2}\end{matrix};-x^2\right]\\
K(k)&=\frac{\pi}{2}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix};k^2\right]\\
E(k)&=\frac{\pi}{2}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},-\frac{1}{2}\\1\end{matrix};k^2\right]
\end{aligned}$$

Theorem 11.13 (二項定理)**.**

$${}_1F_0\left[\begin{matrix}a\\-\end{matrix};x\right]=(1-x)^{-a}$$

Theorem 11.14 (Vandermonde の恒等式)**.**

$$
{}_2F_1\left[\begin{matrix}a,-n\\b\end{matrix};1\right]=\frac{(b-a)_n}{(b)_n}
$$

$$
\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}=\binom{n+m}{r}
$$

Theorem 11.15 (Gauss の超幾何定理)**.**

$$
{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}
$$

Theorem 11.16.

$$
{}_2F_1\left[\begin{matrix}-\frac{n}{2},\frac{1-n}{2}\\\frac{1}{2}+a\end{matrix};1\right]=\frac{2^n(a)_n}{(2a)_n}
$$

Theorem 11.17 (Kummer の定理)**.**

$$
{}_2F_1\left[\begin{matrix}a,b\\1+a-b\end{matrix};-1\right]=\frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma\left(1+a-b\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}
$$

Theorem 11.18 ($x=\frac{1}{2}$ における特殊値)**.**

$$
{}_2F_1\left[\begin{matrix}a,b\\\frac{1+a+b}{2}\end{matrix};\frac{1}{2}\right]=\frac{\sqrt{\pi}\Gamma\left(\frac{1+a+b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}
$$

$$
{}_2F_1\left[\begin{matrix}a,1-a\\b\end{matrix};\frac{1}{2}\right]=\frac{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{1+b-a}{2}\right)}
$$

Theorem 11.19 (Euler 積分表示)**.**

$$
{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};x\right]=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt
$$

Theorem 11.20 (Pfaff's transformation)**.**

$$
{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};x\right]=(1-x)^{-a}\,{}_2F_1\left[\begin{matrix}a,c-b\\c\end{matrix};-\frac{x}{1-x}\right]
$$

Theorem 11.21 (Euler's ransformation)**.**

$$
{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};x\right]=(1-x)^{c-a-b}\,{}_2F_1\left[\begin{matrix}c-a,c-b\\c\end{matrix};x\right]
$$

Theorem 11.22 (Quadratic transformation).

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}a,b\\2b\end{matrix};2x\right] &= (1-x)^{-a}\,{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1+a}{2}\\\frac{1}{2}+b\end{matrix};\frac{x^2}{(1-x)^2}\right]\\
{}_2F_1\left[\begin{matrix}a,b\\1+a-b\end{matrix};x\right] &= (1-x)^{-a}\,{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1+a}{2}-b\\1+a-b\end{matrix};-\frac{4x}{(1-x)^2}\right]\\
{}_2F_1\left[\begin{matrix}2a,2b\\\frac{1}{2}+a+b\end{matrix};=\right]&{}_2F_1\left[\begin{matrix}a,b\\\frac{1}{2}+a+b\end{matrix};4x(1-x)\right],\quad \left(\Re x\le\frac{1}{2}\right)\\
{}_2F_1\left[\begin{matrix}a,b\\1+a-b\end{matrix};x\right] &= (1+x)^{-a}\,{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1+a}{2}\\1+a-b\end{matrix};\frac{4x}{(1+x)^2}\right]\\
{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1+a}{2}\\1+a-b\end{matrix};x\right] &= \left(\frac{2}{1+\sqrt{1-x}}\right)^{a}\,{}_2F_1\left[\begin{matrix}a,b\\1+a-b\end{matrix};\frac{1-\sqrt{1-x}}{1+\sqrt{1+x}}\right]\\
{}_2F_1\left[\begin{matrix}a,b\\2b\end{matrix};\frac{4x}{(1+x)^2}\right] &= (1+x)^{2a}\,{}_2F_1\left[\begin{matrix}a,\frac{1}{2}+a-b\\\frac{1}{2}+b\end{matrix};x^2\right]\\
{}_2F_1\left[\begin{matrix}a,b\\2b\end{matrix};x\right] &= \left(\frac{2}{1+\sqrt{1-x}}\right)^{2a}\,{}_2F_1\left[\begin{matrix}a,\frac{1}{2}+a-b\\\frac{1}{2}+b\end{matrix};\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2\right]
\end{aligned}
$$

Theorem 11.23.

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}a,\frac{1}{2}+a\\2a\end{matrix};x\right] &= \frac{1}{\sqrt{1-x}}\left(\frac{2}{1+\sqrt{1-x}}\right)^{2a-1}\\
{}_2F_1\left[\begin{matrix}a,\frac{1}{2}+a\\1+2a\end{matrix};x\right] &= \left(\frac{2}{1+\sqrt{1-x}}\right)^{2a}
\end{aligned}
$$

11.5　一般化された超幾何関数

Theorem 11.24 (Euler 積分表示).

$$
{}_{r+1}F_{s+1}\left[\begin{matrix}a_1,\dots,a_r,a\\b_1,\dots,b_s,b\end{matrix};x\right]=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\,{}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\b_1,\dots,b_s\end{matrix};xt\right]dt
$$

Theorem 11.25 (Saalschütz の和公式).

$$
{}_3F_2\left[\begin{matrix}a,b,-n\\c,1-n+a+b-c\end{matrix};1\right]=\frac{(c-a,c-b)_n}{(c,c-a-b)_n}
$$

Theorem 11.26 (Dixon の恒等式).

$$
{}_3F_2\left[\begin{matrix}a,b,c\\1+a-b,1+a-c\end{matrix};1\right]=\frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{a}{2}-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}-c\right)\Gamma(1+a-b-c)}
$$

$$
\sum_{n\in\mathbb{Z}}(-1)^n\binom{a+b}{a+n}\binom{b+c}{b+n}\binom{c+a}{c+n}=\frac{(a+b+c)!}{a!b!c!}
$$

Theorem 11.27.

$$
{}_3F_2\left[\begin{matrix}a, 1+\frac{a}{2}, -n\\ \frac{a}{2}, w\end{matrix}; 1\right] = \frac{(w-a-n-1)(w-a)_{n-1}}{(w)_n}
$$
$$
{}_3F_2\left[\begin{matrix}a, b, -n\\ 1+a-b, 1+2b-n\end{matrix}; 1\right] = \frac{\left(a-2b, 1+\frac{a}{2}-b, -b\right)_n}{\left(1+a-b, \frac{a}{2}-b, -2b\right)_n}
$$

Theorem 11.28. $2+a+b=c+d$ としたとき,

$$
{}_3F_2\left[\begin{matrix}1, a, b\\ c, d\end{matrix}; 1\right] = \frac{1}{(c-a-1)(c-b-1)}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} - \frac{(1-c)(1-d)}{(c-a-1)(c-b-1)}
$$

Theorem 11.29.

$$
{}_4F_3\left[\begin{matrix}a, 1+\frac{a}{2}, b, c\\ \frac{a}{2}, 1+a-b, 1+a-c\end{matrix}; 1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+a}{2}-b-c\right)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma\left(\frac{1+a}{2}-b\right)\Gamma\left(\frac{1+a}{2}-c\right)}
$$
$$
{}_4F_3\left[\begin{matrix}a, 1+\frac{a}{2}, b, c\\ \frac{a}{2}, 1+a-b, 1+a-c\end{matrix}; -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}
$$

Theorem 11.30.

$$
{}_4F_3\left[\begin{matrix}a, 1+\frac{a}{2}, b, -n\\ \frac{a}{2}, 1+a-b, 1+2b-n\end{matrix}; 1\right] = \frac{(a-2b, -b)_n}{(1+a-b, -2b)_n}
$$
$$
{}_4F_3\left[\begin{matrix}a, 1+\frac{a}{2}, b, -n\\ \frac{a}{2}, 1+a-b, 2+2b-n\end{matrix}; 1\right] = \frac{\left(a-2b-1, \frac{1+a}{2}-b, -1-b\right)_n}{\left(1+a-b, \frac{a-1}{2}-b, -1-2b\right)_n}
$$

Theorem 11.31 (Clausen's formula). $\frac{1}{2}=b+c+d+n$ としたとき,

$$
{}_4F_3\left[\begin{matrix}-n, b, c, d\\ 1-n-b, 1-n-c, 1-n-d\end{matrix}; 1\right] = \frac{(2b, 2c, b+c)_n}{(b, c, 2b+2c)_n}
$$

$\frac{3}{2}=b+c+d+n$ としたとき,

$$
{}_4F_3\left[\begin{matrix}-n, b, c, d\\ 1-n-b, 1-n-c, 2-n-d\end{matrix}; 1\right] = \frac{\left(2b, 2c, b+c-\frac{1}{2}, b+c\right)_n}{\left(b, c, \frac{1}{2}+b+c, 2b+2c-1\right)_n}
$$

Theorem 11.32.

$$
{}_5F_4\left[\begin{matrix}a, 1+\frac{a}{2}, b, c, d\\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d\end{matrix}; 1\right]
$$
$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}
$$

Theorem 11.33 (Dougall の和公式)**.** $1+2a+n=b+c+d+e$ としたとき,

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}a,1+\frac{a}{2},b,c,d,e,-n\\ \frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-e,1+a+n\end{matrix};1\right]\\
&=\frac{(1+a,1+a-b-c,1+a-b-d,1+a-c-d)_n}{(1+a-b,1+a-c,1+a-d,1+a-b-c-d)_n}
\end{aligned}
$$

Theorem 11.34 (Karlsson-Minton の和公式)**.** n_i を正整数として,

$$\sum_{k=1}^{r}n_k<1+a$$

のとき,

$$
{}_{r+2}F_{r+1}\left[\begin{matrix}-a,b,c_1+n_1,\ldots,c_r+m_r\\ 1+b,c_1,\ldots,c_r\end{matrix};1\right]=\frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(1+a+b)}\prod_{k=1}^{r}\frac{(c_k-b)_{n_k}}{(c_k)_{n_k}}
$$

Theorem 11.35 (Kummer's transformation)**.**

$$
{}_1F_1\left[\begin{matrix}a\\ b\end{matrix};x\right]=e^x{}_1F_1\left[\begin{matrix}b-a\\ b\end{matrix};-x\right]
$$

Theorem 11.36.

$$
{}_1F_1\left[\begin{matrix}a\\ 2a\end{matrix};2x\right]=e^x{}_0F_1\left[\begin{matrix}-\\ \frac{1}{2}+a\end{matrix};\frac{x^2}{4}\right]
$$

Theorem 11.37.

$$
{}_3F_2\left[\begin{matrix}a,b,-n\\ c,d\end{matrix};1\right]=\frac{(d-a)_n}{(d)_n}{}_3F_2\left[\begin{matrix}a,c-b,-n\\ c,1-n+a-d\end{matrix};1\right]
$$

Theorem 11.38.

$$
{}_3F_2\left[\begin{matrix}a,b,c\\ 1+a-b,1+a-c\end{matrix};-1\right]=2^{-a}\,{}_3F_2\left[\begin{matrix}1+a-b-c,\frac{a}{2},\frac{1+a}{2}\\ 1+a-b,1+a-c\end{matrix};1\right]
$$

Theorem 11.39.

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}a,b,-n\\ 1+a-b,w\end{matrix};1\right]&=\frac{(w-a)_n}{(w)_n}\,{}_4F_3\left[\begin{matrix}\frac{a}{2},\frac{1+a}{2}-b,1+a-w,-n\\ 1+a-b,\frac{1+a-w-n}{2},1+\frac{a-w-n}{2}\end{matrix};1\right]\\
{}_3F_2\left[\begin{matrix}a,b,-n\\ 1+a-b,w\end{matrix};-1\right]&=\frac{(w-a)_n}{(w)_n}{}_4F_3\left[\begin{matrix}\frac{a}{2},\frac{1+a}{2},1+a-w,-n\\ 1+a-b,\frac{1+a-w-n}{2},1+\frac{a-w-n}{2}\end{matrix};1\right]
\end{aligned}
$$

Theorem 11.40.

$$ {}_3F_2\left[\begin{matrix}a,b,c\\ w-b,w-c\end{matrix};-1\right]=\frac{\Gamma(w-b)\Gamma(w-c)}{\Gamma(w)\Gamma(w-b-c)}\,{}_4F_3\left[\begin{matrix}\frac{w-a}{2},\frac{1+w-a}{2},b,c\\ w-a,\frac{w}{2},\frac{1+w}{2}\end{matrix};1\right] $$

Theorem 11.41 (Non-terminating Saalschützの和公式)**.** $1+a+b+c=d+e$ としたとき,

$$ \begin{aligned}{}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};1\right]&=\frac{\Gamma(e)\Gamma(1+a-d)\Gamma(1+b-d)\Gamma(1+c-d)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(1-d)}\\ &-\frac{\Gamma(e)\Gamma(1+a-d)\Gamma(1+b-d)\Gamma(1+c-d)\Gamma(d-1)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1+e-d)\Gamma(1-d)}\\ &\times{}_3F_2\left[\begin{matrix}1+a-d,1+b-d,1+c-d\\ 2-d,1+e-d\end{matrix};1\right]\end{aligned} $$

Theorem 11.42 (Quadratic transformation)**.**

$$ {}_3F_2\left[\begin{matrix}a,b,c\\ 1+a-b,1+a-c\end{matrix};x\right]=(1-x)^{-a}\,{}_3F_2\left[\begin{matrix}1+a-b-c,\frac{a}{2},\frac{1+a}{2}\\ 1+a-b,1+a-c\end{matrix};-\frac{4x}{(1-x)^2}\right] $$

Theorem 11.43 (Whipple's transformation)**.** $1+a+b+c=d+e+f+n$ としたとき,

$$ {}_4F_3\left[\begin{matrix}a,b,c,-n\\ d,e,f\end{matrix};1\right]=\frac{(e-a,f-a)_n}{(e,f)_n}\,{}_4F_3\left[\begin{matrix}a,d-b,d-c,-n\\ d,1-n+a-e,1-n+a-f\end{matrix};1\right] $$

Theorem 11.44.

$$ \begin{aligned}&{}_4F_3\left[\begin{matrix}a,b,c,d\\ 1+a-b,1+a-c,1+a-d\end{matrix};-1\right]\\ &=\frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-d\right)}\,{}_3F_2\left[\begin{matrix}1+a-b-c,\frac{a}{2},d\\ 1+a-b,1+a-c\end{matrix};1\right]\end{aligned} $$

Theorem 11.45.

$$ \begin{aligned}&{}_4F_3\left[\begin{matrix}a,b,c,-n\\ 1+a-b,1+a-c,w\end{matrix};1\right]\\ &=\frac{(w-a)_n}{(w)_n}\,{}_5F_4\left[\begin{matrix}1+a-b-c,\frac{a}{2},\frac{1+a}{2},1+a-w,-n\\ 1+a-b,1+a-c,\frac{1+a-w-n}{2},1+\frac{a-w-n}{2}\end{matrix};1\right]\end{aligned} $$

$$ {}_4F_3\left[\begin{matrix}a,1+\frac{a}{2},b,-n\\ \frac{a}{2},1+a-b,w\end{matrix};1\right]=\frac{(w-a)_n}{(w)_n}\,{}_4F_3\left[\begin{matrix}\frac{a}{2}-b,\frac{1+a}{2},1+a-w,-n\\ 1+a-b,\frac{1+a-w-n}{2},1+\frac{a-w-n}{2}\end{matrix};1\right] $$

$$ \begin{aligned}&{}_4F_3\left[\begin{matrix}-n,b,c,d\\ 1-n-b,1-n-c,w\end{matrix};1\right]\\ &=\frac{(w-d)_n}{(w)_n}\,{}_5F_4\left[\begin{matrix}d,1-n-b-c,1-n-w,-\frac{n}{2},\frac{1-n}{2}\\ 1-n-b,1-n-c,\frac{1+d-w-n}{2},1+\frac{d-w-n}{2}\end{matrix};1\right]\end{aligned} $$

Theorem 11.46.

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}a,b,c,-n\\ w-b,w-c,w+n\end{matrix};1\right]\\
&=\frac{(w,w-b-c)_n}{(w-b,w-c)_n}\,{}_5F_4\left[\begin{matrix}\frac{w-a}{2},\frac{1+w-a}{2},b,c,-n\\ w-a,\frac{w}{2},\frac{1+w}{2},1-n+b+c-w\end{matrix};1\right]\\
&{}_4F_3\left[\begin{matrix}a,1+\frac{w}{2},b,c\\ \frac{w}{2},1+w-b,1+w-c\end{matrix};-1\right]\\
&=\frac{\Gamma(1+w-b)\Gamma(1+w-c)}{\Gamma(1+w)\Gamma(1+w-b-c)}\,{}_4F_3\left[\begin{matrix}\frac{w-a}{2},\frac{1+w-a}{2},b,c\\ 1+w-a,\frac{w}{2},\frac{1+w}{2}\end{matrix};1\right]
\end{aligned}
$$

Theorem 11.47 (Quadratic transformation)**.**

$$
{}_4F_3\left[\begin{matrix}a,1+\frac{a}{2},b,c\\ \frac{a}{2},1+a-b,1+a-c\end{matrix};x\right]=\frac{1+x}{(1-x)^{1+a}}\,{}_3F_2\left[\begin{matrix}1+a-b-c,\frac{1+a}{2},1+\frac{a}{2}\\ 1+a-b,1+a-c\end{matrix};-\frac{4x}{(1-x)^2}\right]
$$

Theorem 11.48.

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}a,b,c,d,-n\\ 1+a-b,1+a-c,1+a-d,1+a+n\end{matrix};1\right]\\
&=\frac{\left(1+a,1+\frac{a}{2}-d\right)_n}{\left(1+\frac{a}{2},1+a-d\right)_n}\,{}_4F_3\left[\begin{matrix}1+a-b-c,\frac{a}{2},d,-n\\ 1+a-b,1+a-c,d-n-\frac{a}{2}\end{matrix};1\right]
\end{aligned}
$$

Theorem 11.49.

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}a,1+\frac{a}{2},b,c,-n\\ \frac{a}{2},1+a-b,1+a-c,w\end{matrix};1\right]\\
&=\frac{(w-a-n-1)(w-a)_{n-1}}{(w)_n}\,{}_5F_4\left[\begin{matrix}1+a-b-c,\frac{1+a}{2},1+\frac{a}{2},1+a-w-n\\ 1+a-b,1+a-c,1+\frac{a-w-n}{2},1+\frac{1+a-w-n}{2}\end{matrix};1\right]
\end{aligned}
$$

Theorem 11.50. $\lambda=1+2a-b-c-d$ としたとき,

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix}a,b,c,d,-n\\ 1+a-b,1+a-c,1+a-d,2a-2\lambda-n\end{matrix};1\right]=\frac{(1+\lambda-a,1+2\lambda-a)_n}{(1+\lambda,1+2\lambda-2a)_n}\\
&\times{}_9F_8\left[\begin{matrix}\lambda,1+\frac{\lambda}{2},\frac{a}{2},\frac{1+a}{2},b+\lambda-a,c+\lambda-a,d+\lambda-a,1+2\lambda-a+n,-n\\ \frac{\lambda}{2},\lambda+\frac{1-a}{2},1+\lambda-\frac{a}{2},1+a-b,1+a-c,1+a-d,a-\lambda-n,1+\lambda+n\end{matrix};1\right]\\
&{}_5F_4\left[\begin{matrix}a,b,c,d,-n\\ 1+a-b,1+a-c,1+a-d,1+2a-2\lambda-n\end{matrix};1\right]\\
&=\frac{(\lambda-a)_n(1+2\lambda-a)_{n-1}(2\lambda+2n-a)}{(1+\lambda,2\lambda-2a)_n}\\
&\times{}_9F_8\left[\begin{matrix}\lambda,1+\frac{\lambda}{2},\frac{a}{2},\frac{1+a}{2},b+\lambda-a,c+\lambda-a,d+\lambda-a,2\lambda-a+n,-n\\ \frac{\lambda}{2},\lambda+\frac{1-a}{2},1+\lambda-\frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-\lambda-n,1+\lambda+n\end{matrix};1\right]
\end{aligned}
$$

Theorem 11.51.

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix} a, 1+\frac{w}{2}, b, c, -n \\ \frac{w}{2}, 1+w-b, 1+w-c, 1+w+n \end{matrix}; 1\right] \\
&= \frac{(1+w, 1+w-b-c)_n}{(1+w-b, 1+w-c)_n}\, {}_5F_4\left[\begin{matrix} \frac{w-a}{2}, \frac{1+w-a}{2}, b, c, -n \\ 1+w-a, \frac{w}{2}, \frac{1+w}{2}, b+c-w-n \end{matrix}; 1\right]
\end{aligned}
$$

Theorem 11.52.

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d, e \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e \end{matrix}; -1\right] \\
&= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\, {}_3F_2\left[\begin{matrix} 1+a-b-c, d, e \\ 1+a-b, 1+a-c \end{matrix}; 1\right]
\end{aligned}
$$

Theorem 11.53. $\lambda = 1+2a-b-c-d$ としたとき,

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d, -n \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+2a-2\lambda-n \end{matrix}; 1\right] = \frac{(\lambda-a, 2\lambda-a)_n}{(1+\lambda, 2\lambda-2a)_n} \\
&\times {}_9F_8\left[\begin{matrix} \lambda, 1+\frac{\lambda}{2}, \frac{1+a}{2}, 1+\frac{a}{2}, , b+\lambda-a, c+\lambda-a, d+\lambda-a, 2\lambda-a+n, -n \\ \frac{\lambda}{2}, \lambda-\frac{a}{2}, \lambda+\frac{1-a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-\lambda-n, 1+\lambda+n \end{matrix}; 1\right] \\
&{}_6F_5\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d, -n \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 2+2a-2\lambda-n \end{matrix}; 1\right] \\
&= \frac{(\lambda-a-1)_n(2\lambda-a)_{n-1}(2\lambda+2n-a-1)_n}{(1+\lambda, 2\lambda-2a-1)_n} \\
&\times {}_9F_8\left[\begin{matrix} \lambda, 1+\frac{\lambda}{2}, \frac{1+a}{2}, 1+\frac{a}{2}, , b+\lambda-a, c+\lambda-a, d+\lambda-a, 2\lambda-a+n-1, -n \\ \frac{\lambda}{2}, \lambda-\frac{a}{2}, \lambda+\frac{1-a}{2}, 1+a-b, 1+a-c, 1+a-d, 2+a-\lambda-n, 1+\lambda+n \end{matrix}; 1\right]
\end{aligned}
$$

Theorem 11.54 (Whipple's transformation)**.**

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, b, c, d, e, -n \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n \end{matrix}; 1\right] \\
&= \frac{(1+a, 1+a-d-e)_n}{(1+a-d, 1+a-e)_n}\, {}_4F_3\left[\begin{matrix} 1+a-b-c, d, e, -n \\ 1+a-b, 1+a-c, d+e-n-a \end{matrix}; 1\right]
\end{aligned}
$$

Theorem 11.55 (Non-terminating Whipple's transformation)**.**

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix}a,1+\frac{a}{2},b,c,d,e,f\\ \frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-e,1+a-f\end{matrix};1\right]\\
&=\frac{\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-d-e-f)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-d-f)\Gamma(1+a-e-f)}\\
&\times{}_4F_3\left[\begin{matrix}1+a-b-c,d,e,f\\ 1+a-b,1+a-c,d+e+f-a\end{matrix};1\right]\\
&+\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(d)\Gamma(e)\Gamma(f)}\\
&\times\frac{\Gamma(2+2a-b-c-d-e-f)\Gamma(d+e+f-a-1)}{\Gamma(2+2a-b-d-e-f)\Gamma(2+2a-c-d-e-f)}\\
&\times{}_4F_3\left[\begin{matrix}2+2a-b-c-d-e-f,1+a-d-e,1+a-d-f,1+a-e-f\\ 2+a-d-e-f,2+2a-b-d-e-f,2+2a-c-d-e-f\end{matrix};1\right]
\end{aligned}
$$

Theorem 11.56.

$$\lambda=1+2a-b-c-d,\ 2+3a+n=b+c+d+e+f+g$$

としたとき,

$$
\begin{aligned}
&{}_9F_8\left[\begin{matrix}a,1+\frac{a}{2},b,c,d,e,f,g,-n\\ \frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-e,1+a-f,1+a-g,1+a+n\end{matrix};1\right]\\
&=\frac{(1+a,1+\lambda-e,1+\lambda-f,1+\lambda-g)_n}{(1+\lambda,1+a-e,1+a-f,1+a-g)_n}\\
&\times{}_9F_8\left[\begin{matrix}\lambda,1+\frac{\lambda}{2},b+\lambda-a,c+\lambda-a,d+\lambda-a,e,f,g,-n\\ \frac{\lambda}{2},1+a-b,1+a-c,1+a-d,1+\lambda-e,1+\lambda-f,1+\lambda-g,1+\lambda+n\end{matrix};1\right]
\end{aligned}
$$

Theorem 11.57 (Expansion formula)**.**

$$
\begin{aligned}
&{}_{r+4}F_{r+3}\left[\begin{matrix}a,b,c,a_1,\dots,a_r,-n\\ 1+a-b,1+a-c,b_1,\dots,b_{r+1}\end{matrix};x\right]\\
&=\sum_{k=0}^{n}\frac{(1+a-b-c,a_1,\dots,a_r,-n)_k(a)_{2k}}{(1+a-b,1+a-c,b_1,\dots,b_{r+1})_k k!}(-x)^k\\
&\times{}_{r+2}F_{r+1}\left[\begin{matrix}a+2k,a_1+k,\dots,a_r+k,k-n\\ b_1+k,\dots,b_{r+1}+k\end{matrix};x\right]
\end{aligned}
$$

Theorem 11.58.

$$
\begin{aligned}
{}_0F_1\left[\begin{matrix}-\\ a\end{matrix};x\right]{}_0F_1\left[\begin{matrix}-\\ b\end{matrix};x\right]&={}_2F_3\left[\begin{matrix}\frac{a+b-1}{2},\frac{a+b}{2}\\ a,b,a+b-1\end{matrix};4x\right]\\
{}_0F_1\left[\begin{matrix}-\\ a\end{matrix};x\right]{}_0F_1\left[\begin{matrix}-\\ a\end{matrix};-x\right]&={}_0F_3\left[\begin{matrix}-\\ a,\frac{a}{2},\frac{1+a}{2}\end{matrix};-\frac{x^2}{4}\right]
\end{aligned}
$$

Theorem 11.59.

$$
{}_1F_1\left[\begin{matrix}a\\b\end{matrix};x\right]{}_1F_1\left[\begin{matrix}a\\b\end{matrix};-x\right]={}_2F_3\left[\begin{matrix}a,b-a\\b,\frac{b}{2},\frac{1+b}{2}\end{matrix};\frac{x^2}{4}\right]
$$

$$
{}_1F_1\left[\begin{matrix}a\\2a\end{matrix};x\right]^2=e^x{}_1F_2\left[\begin{matrix}a\\2a,\frac{1}{2}+a\end{matrix};\frac{x^2}{4}\right]
$$

Theorem 11.60.

$$
{}_0F_2\left[\begin{matrix}-\\a,b\end{matrix};x\right]{}_0F_2\left[\begin{matrix}-\\a,b\end{matrix};-x\right]=\sum_{0\le n}\frac{(-1)^n(a+b-1)_n}{(a,b)_n(a,b,a+b-1)_{2n}n!}x^{2n}
$$

Theorem 11.61 (Clausen's formula)**.**

$$
{}_2F_1\left[\begin{matrix}a,b\\\frac{1}{2}+a+b\end{matrix};x\right]^2={}_3F_2\left[\begin{matrix}2a,2b,a+b\\\frac{1}{2}+a+b,2a+2b\end{matrix};x\right]
$$

$$
{}_2F_1\left[\begin{matrix}a,b\\\frac{1}{2}+a+b\end{matrix};x\right]{}_2F_1\left[\begin{matrix}a,b\\a+b-\frac{1}{2}\end{matrix};x\right]={}_3F_2\left[\begin{matrix}2a,2b,a+b\\\frac{1}{2}+a+b,2a+2b-1\end{matrix};x\right]
$$

$$
{}_2F_1\left[\begin{matrix}a,b\\a+b-\frac{1}{2}\end{matrix};x\right]{}_2F_1\left[\begin{matrix}a,b-1\\a+b-\frac{1}{2}\end{matrix};x\right]={}_3F_2\left[\begin{matrix}2a,2b-1,a+b-1\\a+b-\frac{1}{2},2a+2b-2\end{matrix};x\right]
$$

$$
{}_2F_1\left[\begin{matrix}a,b\\\frac{1+a+b}{2}\end{matrix};x\right]^2={}_3F_2\left[\begin{matrix}a,b,\frac{a+b}{2}\\a+b,\frac{1+a+b}{2}\end{matrix};4x(1-x)\right],\quad\left(\Re x\le\frac{1}{2}\right)
$$

11.6 Bilateral 超幾何級数

Definition 11.62.

$$
{}_rH_r\left[\begin{matrix}a_1,\dots,a_r\\b_1,\dots,b_r\end{matrix};x\right]:=\sum_{n=-\infty}^{\infty}\frac{(a_1,\dots,a_r)_n}{(b_1,\dots,b_r)_n}x^n
$$

Theorem 11.63.

$$
{}_1H_1\left[\begin{matrix}a\\c\end{matrix};1\right]=0
$$

Theorem 11.64 (Dougall の和公式)**.**

$$
{}_2H_2\left[\begin{matrix}a,b\\c,d\end{matrix};1\right]=\frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}
$$

Theorem 11.65 (Bailey の和公式)**.**

$$
\lambda=\frac{(f-a)(f-b)-(1+f-c)(1+f-d)}{f}
$$

としたとき,

$$ {}_3H_3\left[\begin{matrix} a, b, 1+f \\ c, d, f \end{matrix}; 1\right] = \lambda \frac{\Gamma(c)\Gamma(d)\Gamma(1-a)\Gamma(1-b)\Gamma(c+d-a-b-2)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)} $$

Theorem 11.66.

$$ \begin{aligned} &{}_3H_3\left[\begin{matrix} b, c, d \\ 1+a-b, 1+a-c, 1+a-d \end{matrix}; 1\right] \\ &= \cos\frac{\pi a}{2} \frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \\ &\times \frac{\Gamma\left(1+\frac{3a}{2}-b-c-d\right)}{\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}-c\right)\Gamma\left(1+\frac{a}{2}-d\right)} \end{aligned} $$

Theorem 11.67.

$$ \begin{aligned} &{}_4H_4\left[\begin{matrix} 1+\frac{a}{2}, b, c, d \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d \end{matrix}; 1\right] \\ &= \frac{2}{a}\sin\frac{\pi a}{2} \frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \\ &\times \frac{\Gamma\left(\frac{1+3a}{2}-b-c-d\right)}{\Gamma\left(\frac{1+a}{2}-b\right)\Gamma\left(\frac{1+a}{2}-c\right)\Gamma\left(\frac{1+a}{2}-d\right)} \\ &{}_4H_4\left[\begin{matrix} 1+\frac{a}{2}, b, c, d \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d \end{matrix}; -1\right] \\ &= \frac{\sin\pi a}{\pi a} \frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \end{aligned} $$

Theorem 11.68.

$$ \begin{aligned} &{}_5H_5\left[\begin{matrix} 1+\frac{a}{2}, b, c, d, e \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e \end{matrix}; 1\right] \\ &= \frac{\sin\pi a}{\pi a} \frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(1-e)}{\Gamma(1+a-b-c)\Gamma(1+a-b-d)} \\ &\times \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+2a-b-c-d-e)}{\Gamma(1+a-b-e)\Gamma(1+a-c-d)\Gamma(1+a-c-e)\Gamma(1+a-d-e)} \end{aligned} $$

Theorem 11.69.

$$
\begin{aligned}
&{}_6H_6\left[\begin{matrix}1+\frac{a}{2},b,c,d,e,f\\ \frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-e,1+a-f\end{matrix};-1\right]\\
&=\frac{\sin\pi a}{\pi a}\frac{\Gamma(1-b)\Gamma(1-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a-b-c)}\\
&\times\Bigg(\frac{\Gamma(1+a-d-e-f)}{\Gamma(1+a-d-e)\Gamma(1+a-d-f)\Gamma(1+a-e-f)}\\
&\times{}_3H_3\left[\begin{matrix}d,e,f\\ 1+a-b,1+a-c,d+e+f-a\end{matrix};1\right]\\
&+\frac{\pi}{\sin\pi(d+e+f-a)}\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(d)\Gamma(e)\Gamma(f)}\\
&\times\frac{1}{\Gamma(2+2a-b-d-e-f)\Gamma(2+2a-c-d-e-f)}\\
&\times{}_3F_2\left[\begin{matrix}1+a-d-e,1+a-d-f,1+a-e-f\\ 2+2a-b-d-e-f,2+2a-c-d-e-f\end{matrix};1\right]\Bigg)
\end{aligned}
$$

11.7 Appell 級数

Definition 11.70.

$$
\begin{aligned}
F_1\left[\begin{matrix}a;b_1,b_2\\ c\end{matrix};x,y\right]&:=\sum_{0\le n,m}\frac{(a)_{n+m}(b_1)_n(b_2)_m}{(c)_{n+m}n!m!}x^ny^m\\
F_2\left[\begin{matrix}a;b_1,b_2\\ c_1,c_2\end{matrix};x,y\right]&:=\sum_{0\le n,m}\frac{(a)_{n+m}(b_1)_n(b_2)_m}{(c_1)_n(c_2)_mn!m!}x^ny^m\\
F_3\left[\begin{matrix}a_1,a_2;b_1,b_2\\ c\end{matrix};x,y\right]&:=\sum_{0\le n,m}\frac{(a_1,b_1)_n(b_1,b_2)_m}{(c)_{n+m}n!m!}x^ny^m\\
F_4\left[\begin{matrix}a,b\\ c_1,c_2\end{matrix};x,y\right]&:=\sum_{0\le n,m}\frac{(a,b)_{n+m}}{(c_1)_n(c_2)_mn!m!}x^ny^m
\end{aligned}
$$

Theorem 11.71.

$$
\begin{aligned}
F_1\left[\begin{matrix}a;b,b\\ 1+a-b\end{matrix};e^{2\pi i/3},e^{-2\pi i/3}\right]&=\frac{\Gamma\left(1+\frac{a}{3}\right)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{3}-b\right)}\\
F_1\left[\begin{matrix}1-a;a,a\\ b\end{matrix};\frac{e^{i\pi/6}}{\sqrt{3}},\frac{e^{-i\pi/6}}{\sqrt{3}}\right]&=3^a\frac{\Gamma(b)\Gamma\left(\frac{2+a+b}{3}\right)}{\Gamma(a+b)\Gamma\left(\frac{2-2a+b}{3}\right)}
\end{aligned}
$$

Theorem 11.72.

$$F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,x\right]={}_2F_1\left[\begin{matrix}a,b_1+b_2\\c\end{matrix};x\right]$$
$$F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};1,x\right]=\frac{\Gamma(c)\Gamma(c-a-b_1)}{\Gamma(c-a)\Gamma(c-b_1)}\,{}_2F_1\left[\begin{matrix}a,b_2\\c-b_1\end{matrix};x\right]$$
$$F_1\left[\begin{matrix}a;b,c-b\\c\end{matrix};x,y\right]=(1-y)^{-a}{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};\frac{x-y}{1-y}\right]$$

Theorem 11.73.

$$F_3\left[\begin{matrix}a,c-a;b,c-b\\c\end{matrix};x,y\right]=(1-y)^{a+b-c}{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};x+y-xy\right]$$

Theorem 11.74.

$$F_4\left[\begin{matrix}a,b\\c_1,c_2\end{matrix};x,x\right]={}_4F_3\left[\begin{matrix}a,b,\frac{c_1+c_2-1}{2},\frac{c_1+c_2}{2}\\c_1,c_2,c_1+c_2-1\end{matrix};4x\right]$$

Theorem 11.75 (積分表示)**.**

$$F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,y\right]=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-xt)^{-b_2}\,dt$$
$$F_2\left[\begin{matrix}a;b_1,b_2\\c_1,c_2\end{matrix};x,y\right]=\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1)\Gamma(c_2-b_2)}$$
$$\times\int_0^1\int_0^1 s^{b_1-1}(1-s)^{c_1-b_1-1}t^{b_2-1}(1-t)^{c_2-b_2-1}(1-xs-yt)^{-a}\,dsdt$$
$$F_3\left[\begin{matrix}a_1,a_2;b_1,b_2\\c\end{matrix};x,y\right]=\frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c-a_1-a_2)}$$
$$\times\int_{0<s+t<1} s^{a_1-1}t^{a_2-1}(1-xs)^{-b_1}(1-yt)^{-b_2}(1-s-t)^{c-a_1-a_2-1}\,dsdt$$

Theorem 11.76 (F_1 の変換公式)**.**

$$F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,y\right]=(1-x)^{-b_1}(1-y)^{-b_2}F_1\left[\begin{matrix}c-a;b_1,b_2\\c\end{matrix};-\frac{x}{1-x},-\frac{y}{1-y}\right]$$
$$F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,y\right]=(1-x)^{c-a-b_1}(1-y)^{-b_2}F_1\left[\begin{matrix}c-a;c-b_1-b_2,b_2\\c\end{matrix};x,\frac{x-y}{1-y}\right]$$
$$F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,y\right]=(1-x)^{-b_1}(1-y)^{c-a-b_2}F_1\left[\begin{matrix}c-a;b_1,c-b_1-b_2\\c\end{matrix};\frac{y-x}{1-x},y\right]$$
$$F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,y\right]=(1-x)^{-a}F_1\left[\begin{matrix}a;c-b_1-b_2,b_2\\c\end{matrix};-\frac{x}{1-x},\frac{y-x}{1-x}\right]$$
$$F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,y\right]=(1-y)^{-a}F_1\left[\begin{matrix}a;b_1,c-b_1-b_2\\c\end{matrix};\frac{x-y}{1-y},-\frac{y}{1-y}\right]$$

Theorem 11.77 (F_2 の変換公式)**.**

$$F_2\left[\begin{matrix}a;b_1,b_2\\c_1,c_2\end{matrix};x,y\right]=(1-x)^{-a}F_2\left[\begin{matrix}a,c_1-b_1,b_2\\c_1,c_2\end{matrix};-\frac{x}{1-x},\frac{y}{1-x}\right]$$
$$F_2\left[\begin{matrix}a;b_1,b_2\\c_1,c_2\end{matrix};x,y\right]=(1-y)^{-a}F_2\left[\begin{matrix}a;b_1,c_2-b_2\\c_1,c_2\end{matrix};\frac{x}{1-y},-\frac{y}{1-y}\right]$$
$$F_2\left[\begin{matrix}a;b_1,b_2\\c_1,c_2\end{matrix};x,y\right]=(1-x-y)^{-a}F_2\left[\begin{matrix}a;c_1-b_1,c_2-b_2\\c_1,c_2\end{matrix};-\frac{x}{1-x-y},-\frac{y}{1-x-y}\right]$$

Theorem 11.78 (F_3 の変換公式)**.**

$$F_3\left[\begin{matrix}a,c-a;b_1,b_2\\c\end{matrix};x,y\right]=(1-y)^{-b_2}F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,-\frac{y}{1-y}\right]$$

Theorem 11.79 (Lauricella 超幾何級数)**.**

$$(\boldsymbol{a})_{\boldsymbol{n}}:=(a_1)_{n_1}\cdots(a_r)_{n_r}$$

とする.

$$F_A^{(r)}\left[\begin{matrix}a;\boldsymbol{b}\\\boldsymbol{c}\end{matrix};\boldsymbol{x}\right]=\sum_{0\le\boldsymbol{n}}\frac{(a)_{n_1+\cdots,n_r}(\boldsymbol{b})_{\boldsymbol{n}}}{(\boldsymbol{c})_{\boldsymbol{n}}n_1!\cdots n_r!}\boldsymbol{x}^{\boldsymbol{n}}$$
$$F_B^{(r)}\left[\begin{matrix}\boldsymbol{a};\boldsymbol{b}\\c\end{matrix};\boldsymbol{x}\right]=\sum_{0\le\boldsymbol{n}}\frac{(\boldsymbol{a})_{\boldsymbol{n}}(\boldsymbol{b})_{\boldsymbol{n}}}{(c)_{n_1+\cdots+n_r}n_1!\cdots n_r!}\boldsymbol{x}^{\boldsymbol{n}}$$
$$F_C^{(r)}\left[\begin{matrix}a;b\\\boldsymbol{c}\end{matrix};\boldsymbol{x}\right]=\sum_{0\le\boldsymbol{n}}\frac{(a)_{n_1+\cdots+n_r}(b)_{n_1+\cdots+n_r}}{(\boldsymbol{c})_{\boldsymbol{n}}n_1!\cdots n_r!}\boldsymbol{x}^{\boldsymbol{n}}$$
$$F_D^{(r)}\left[\begin{matrix}a;\boldsymbol{b}\\c\end{matrix};\boldsymbol{x}\right]=\sum_{0\le\boldsymbol{n}}\frac{(a)_{n_1+\cdots+n_r}(\boldsymbol{b})_{\boldsymbol{n}}}{(c)_{n_1+\cdots+n_r}n_1!\cdots n_r!}\boldsymbol{x}^{\boldsymbol{n}}$$

Theorem 11.80.

$$F_A^{(2)}\left[\begin{matrix}a;b_1,b_2\\c_1,c_2\end{matrix};x,y\right]=F_2\left[\begin{matrix}a;b_1,b_2\\c_1,c_2\end{matrix};x,y\right]$$
$$F_B^{(2)}\left[\begin{matrix}a_1,a_2;b_1,b_2\\c\end{matrix};x,y\right]=F_3\left[\begin{matrix}a_1,a_2;b_1,b_2\\c\end{matrix};x,y\right]$$
$$F_C^{(2)}\left[\begin{matrix}a;b\\c_1,c_2\end{matrix};x,y\right]=F_4\left[\begin{matrix}a;b\\c_1,c_2\end{matrix};x,y\right]$$
$$F_D^{(2)}\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,y\right]=F_1\left[\begin{matrix}a;b_1,b_2\\c\end{matrix};x,y\right]$$

Theorem 11.81.

$$F_D^{(r+1)}\left[\begin{matrix}a;\boldsymbol{b},b\\c\end{matrix};\boldsymbol{x},1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,F_D^{(r)}\left[\begin{matrix}a;\boldsymbol{b}\\c-b\end{matrix};\boldsymbol{x}\right]$$

Theorem 11.82 (積分表示)**.**

$$F_D^{(r)}\left[\begin{matrix}a;\boldsymbol{b}\\c\end{matrix};\boldsymbol{x}\right]=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\prod_{i=1}^{r}(1-x_it)^{-b_i}\,dt$$

12 q 特殊関数

12.1 q-Pochhammer 記号

12.2 q ガンマ関数

Definition 12.1.

$$\Gamma_q(x):=(1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}$$

Theorem 12.2 (乗法定理)**.**

$$\prod_{k=0}^{n-1}\Gamma_{q^n}\left(x+\frac{k}{n}\right)=\left(\frac{1-q^n}{1-q}\right)^{1-nx}\Gamma_q(nx)\prod_{k=1}^{n-1}\Gamma_{q^n}\left(\frac{k}{n}\right)$$

12.3 q ベータ関数

Definition 12.3.

$$B_q(x,y):=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$$

13 q 超幾何関数

13.1 q 超幾何関数

Definition 13.1.

$${}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\b_1,\dots,b_s\end{matrix};q;x\right]:=\sum_{0\le n}\frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n}\left((-1)^nq^{\binom{n}{2}}\right)^{1+s-r}x^n$$

底が q である場合は,

$$ {}_r\phi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; x\right] := {}_r\phi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q; x\right] $$

と略記する.

Theorem 13.2 (q 二項定理)**.**

$$ {}_1\phi_0 \left[\begin{matrix} a \\ - \end{matrix}; x\right] = \frac{(ax;q)_\infty}{(x;q)_\infty} $$

Theorem 13.3.

$$ {}_0\phi_1 \left[\begin{matrix} - \\ x \end{matrix}; x\right] = \frac{1}{(x;q)_\infty} $$

Theorem 13.4.

$$ {}_1\phi_1 \left[\begin{matrix} a \\ b \end{matrix}; \frac{b}{a}\right] = \frac{(b/a;q)_\infty}{(b;q)_\infty} $$

Theorem 13.5.

$$ {}_1\phi_1 \left[\begin{matrix} a \\ 0 \end{matrix}; -q\right] = (-q;q)_\infty (aq;q^2)_\infty $$

Theorem 13.6 (q-Vandermonde の恒等式)**.**

$$ {}_2\phi_1 \left[\begin{matrix} a, q^{-n} \\ b \end{matrix}; \frac{bq^{-n}}{a}\right] = \frac{(b/a;q)_n}{(b;q)_n} $$

$$ {}_2\phi_1 \left[\begin{matrix} a, q^{-n} \\ b \end{matrix}; q\right] = \frac{(b/a;q)_n}{(b;q)_n} a^n $$

Theorem 13.7 (Heine の和公式)**.**

$$ {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; \frac{c}{ab}\right] = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty} $$

Theorem 13.8 (Bailey-Daum の和公式)**.**

$$ {}_2\phi_1 \left[\begin{matrix} a, b \\ aq/b \end{matrix}; -\frac{q}{b}\right] = \frac{(-q;q)_\infty (aq, aq^2/b^2;q^2)_\infty}{(aq/b, -q/b;q)_\infty} $$

Theorem 13.9.

$$ {}_2\phi_2 \left[\begin{matrix} a, q/a \\ b, -q \end{matrix}; -b\right] = \frac{(ab, bq/a;q^2)_\infty}{(b;q)_\infty} $$

$$ {}_2\phi_2 \left[\begin{matrix} a, b \\ \sqrt{abq}, -\sqrt{abq} \end{matrix}; -q\right] = \frac{(aq, bq;q^2)_\infty}{(q, abq;q^2)_\infty} $$

Theorem 13.10 (q-Saalschütz の和公式)**.**

$$
{}_3\phi_2\left[\begin{matrix}a,b,q^{-n}\\ c,abq^{1-n}/c\end{matrix};q\right]=\frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}
$$

Theorem 13.11 (q-Dixon の和公式)**.**

$$
{}_4\phi_3\left[\begin{matrix}a,-\sqrt{a}q,b,c\\ -\sqrt{a},aq/b,aq/c\end{matrix};\frac{\sqrt{a}q}{bc}\right]=\frac{(aq,aq/bc,\sqrt{a}q/b,\sqrt{a}q/c;q)_\infty}{(aq/b,aq/c,\sqrt{a}q,\sqrt{a}q/bc;q)_\infty}
$$

Theorem 13.12.

$$
{}_4\phi_3\left[\begin{matrix}a,\sqrt{a}q,b,c\\ \sqrt{a},aq/b,aq/c\end{matrix};-\frac{\sqrt{a}q}{bc}\right]=\frac{(aq,aq/bc,-\sqrt{a}q/b,-\sqrt{a}q/c;q)_\infty}{(aq/b,aq/c,-\sqrt{a}q,-\sqrt{a}q/bc;q)_\infty}
$$

Theorem 13.13. $|b|>1$ のとき,

$$
{}_4\phi_3\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,b\\ \sqrt{a},-\sqrt{a},aq/b\end{matrix};\frac{1}{b}\right]=0
$$

Theorem 13.14.

$$
{}_5\phi_5\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,b,c\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,0\end{matrix};\frac{aq}{bc}\right]=\frac{(aq,aq/bc;q)_\infty}{(aq/b,aq/c;q)_\infty}
$$

Theorem 13.15.

$$
{}_6\phi_5\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,b,c,d\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d\end{matrix};\frac{aq}{bcd}\right]=\frac{(aq,aq/bc,aq/bd,aq/cd;q)_\infty}{(aq/b,aq/c,aq/d,aq/bcd;q)_\infty}
$$

Theorem 13.16 (Jackson の和公式)**.** $a^2q^{1+n}=bcde$ としたとき,

$$
{}_8\phi_7\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,b,c,d,e,q^{-n}\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e,aq^{1+n}\end{matrix};q\right]=\frac{(aq,aq/bc,aq/bd,aq/cd;q)_n}{(aq/b,aq/c,aq/d,aq/bcd;q)_n}
$$

Theorem 13.17 (q-Karlsson-Minton の和公式)**.**

$$
{}_{r+2}\phi_{r+1}\left[\begin{matrix}a,b,c_1q^{n_1},\dots,c_rq^{n_r}\\ bq,c_1,\dots,c_r\end{matrix};\frac{q^{1-n_1-\cdots-n_r}}{a}\right]=\frac{(q,bq/a;q)_\infty}{(q/a,bq;q)_\infty}\prod_{k=1}^{r}\frac{(c_k/b;q)_{n_k}}{(c_k;q)_{n_k}}b^{n_k}
$$

$$
{}_{r+1}\phi_r\left[\begin{matrix}a,c_1q^{n_1},\dots,c_rq^{n_r}\\ c_1,\dots,c_r\end{matrix};\frac{q^{-n_1-\cdots-n_r}}{a}\right]=0,\quad \left(|q^{-n_1-\cdots-n_r}|<|a|\right)
$$

Theorem 13.18.

$$
{}_1\phi_1\left[\begin{matrix}b/a\\ b\end{matrix};x\right]=(x;q)_\infty\,{}_1\phi_2\left[\begin{matrix}a\\ b,x\end{matrix};\frac{bx}{a}\right]
$$

$$
{}_1\phi_1\left[\begin{matrix}0\\ a\end{matrix};x\right]=(x;q)_\infty\,{}_0\phi_2\left[\begin{matrix}-\\ a,x\end{matrix};ax\right]
$$

$$
(b;q)_\infty\,{}_1\phi_1\left[\begin{matrix}b/a\\ b\end{matrix};c\right]=(c;q)_\infty\,{}_1\phi_1\left[\begin{matrix}c/a\\ c\end{matrix};b\right]
$$

Theorem 13.19 (Heine's transformation)**.**

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};x\right] &= \frac{(b,ax;q)_\infty}{(c,x;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/b,x\\ax\end{matrix};b\right]\\
&= \frac{(c/b,bx;q)_\infty}{(c,x;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}abx/c,b\\bx\end{matrix};\frac{c}{b}\right]\\
&= \frac{(abx/c;q)_\infty}{(x;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}c/a,c/b\\c\end{matrix};\frac{abx}{c}\right]
\end{aligned}
$$

Theorem 13.20 (Jackson's transformation)**.**

$$
{}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};x\right] = \frac{(ax;q)_\infty}{(x;q)_\infty}\,{}_2\phi_2\left[\begin{matrix}a,c/b\\c,ax\end{matrix};bx\right]
$$

Theorem 13.21.

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a,0\\b\end{matrix};x\right] &= \frac{1}{(x;q)_\infty}\,{}_1\phi_1\left[\begin{matrix}b/a\\b\end{matrix};ax\right]\\
{}_2\phi_1\left[\begin{matrix}0,0\\a\end{matrix};x\right] &= \frac{1}{(x;q)_\infty}\,{}_0\phi_1\left[\begin{matrix}-\\a\end{matrix};ax\right]
\end{aligned}
$$

Theorem 13.22.

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix}a,q^{-n}\\b\end{matrix};x\right] &= \frac{(b/a;q)_n}{(b;q)_n}\,{}_3\phi_2\left[\begin{matrix}a,axq^{-n}/b,q^{-n}\\aq^{1-n}/b,0\end{matrix};q\right]\\
{}_2\phi_1\left[\begin{matrix}a,q^{-n}\\b\end{matrix};x\right] &= \frac{(b/a;q)_n}{(b;q)_n}b^n\,{}_3\phi_1\left[\begin{matrix}a,q/x,q^{-n}\\aq^{1-n}/b\end{matrix};\frac{x}{b}\right]\\
{}_2\phi_1\left[\begin{matrix}a,q^{-n}\\b\end{matrix};x\right] &= (axq^{-n}/b;q)_n\,{}_3\phi_2\left[\begin{matrix}b/a,q^{-n},0\\b,bq/ax\end{matrix};q\right]
\end{aligned}
$$

Theorem 13.23.

$$
\begin{aligned}
&{}_7\phi_7\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,b,c,d,e\\\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e,0\end{matrix};\frac{a^2q^2}{bcde}\right]\\
&= \frac{(aq,aq/de;q)_\infty}{(aq/d,aq/e;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}aq/bc,d,e\\aq/b,aq/c\end{matrix};\frac{aq}{bc}\right]
\end{aligned}
$$

Theorem 13.24 (Watson's transformation)**.**

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,b,c,d,e,q^{-n}\\\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e,aq^{1+n}\end{matrix};\frac{a^2q^{2+n}}{bcde}\right]\\
&= \frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,d,e,q^{-n}\\aq/b,aq/c,deq^{-n}/a\end{matrix};q\right]
\end{aligned}
$$

13.2 Bilateral q 超幾何関数

Definition 13.25.

$$ {}_r\psi_r \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}; q; x \right] := \sum_{n=-\infty}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_r; q)_n} x^n $$

底が q のときは,

$$ {}_r\psi_r \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}; x \right] := {}_r\psi_r \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}; q; x \right] $$

と略記する.

Theorem 13.26 (Jacobi の三重積)**.**

$$ (q, x, q/x; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n $$

Theorem 13.27 (Ramanujan の和公式)**.**

$$ {}_1\psi_1 \left[\begin{matrix} a \\ b \end{matrix}; x \right] = \frac{(ax, q, b/a, q/ax; q)_\infty}{(x, b, q/a, b/ax; q)_\infty} $$

14 超越的特殊関数

14.1 Lambert W 関数

Definition 14.1. a についての方程式,

$$ ae^a = x $$

の解を Lambert W 関数といい, $W(x)$ で表す. 制約条件, $x \geq -\frac{1}{e}$, $W(x) \leq -1$ を追加することにより, 一価関数となる.

Theorem 14.2 (特殊値)**.**

$$ \begin{aligned} &W(0) = 0 \\ &W(2\ln 2) = \ln 2 \\ &W\left(-\frac{1}{e}\right) = -1 \\ &W(e) = 1 \\ &W'(0) = 1 \end{aligned} $$

Theorem 14.3 (級数表示)**.**

$$W(x) = \sum_{0<n} \frac{(-n)^{n-1}}{n!} x^n$$
$$\left(\frac{W(x)}{x}\right)^a = a \sum_{0\le n} \frac{(n+a)^{n-1}}{n!} (-x)^n$$

14.2 Mittag-Leffler 関数

Definition 14.4.

$$E_{a,b}(x) := \sum_{0\le n} \frac{x^n}{\Gamma(an+b)}$$

15 数論的関数

15.1 Möbius 関数

Definition 15.1. n が平方因子をもつとき,

$$\mu(n) = 0$$

n が相異なる r 個の素因数の積に分解できるとき,

$$\mu(n) = (-1)^r$$

Theorem 15.2. 互いに素な n, m に対して,

$$\mu(nm) = \mu(n)\mu(m)$$

Theorem 15.3.

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & (n = 1) \\ 0, & (n \neq 1) \end{cases}$$

Theorem 15.4.

$$\mu(n) = \sum_{\substack{0\le k<n \\ (n,k)=1}} e^{2\pi ik/n}$$

Theorem 15.5 (Möbius の反転公式)**.**

$$g(n) = \sum_{d|n} f(d)$$

であるとき,

$$f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right)$$

Theorem 15.6.

$$\frac{1}{\zeta(s)} = \sum_{0<n} \frac{\mu(n)}{n^s}$$

15.2 約数関数

Definition 15.7.

$$\sigma_x(n) := \sum_{d|n} d^x$$
$$\sigma(n) := \sigma_1(n)$$
$$d(n) := \sigma_0(n)$$

Theorem 15.8. n, m が互いに素であるとき,

$$\sigma_x(nm) = \sigma_x(n)\sigma_x(m)$$

Theorem 15.9.

$$\sigma_x(n) = \sum_{0<a\leq b\leq n} b^{x-1} \cos\frac{2\pi na}{b}$$

Theorem 15.10.

$$\zeta(s)\zeta(s-x) = \sum_{0<n} \frac{\sigma_x(n)}{n^s}$$

15.3 Euler の φ 関数

Definition 15.11.

$$\varphi(n) := \sum_{\substack{0\leq k<n \\ (n,k)=1}} 1$$

Theorem 15.12. n の素因数分解を

$$n = \prod_{i=1}^{r} p_i^{e_i}$$

とするとき,

$$\varphi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$$

15.4 Von Mangoldt 関数

Definition 15.13.

$$\Lambda(n) := \begin{cases} \ln p, & \exists p : \text{prime}, \exists r \leq 1, n = p^r \\ 0, & \text{otherwise} \end{cases}$$

Theorem 15.14.

$$\ln n = \sum_{d|n} \Lambda(d)$$
$$\Lambda(n) = -\sum_{d|n} \mu(d) \ln d$$

Theorem 15.15.

$$\ln \zeta(s) = \sum_{1<n} \frac{\Lambda(n)}{n^s \ln n}$$
$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{0<n} \frac{\Lambda(n)}{n^s}$$

15.5 分割数

Definition 15.16.

$$\frac{1}{(q;q)_\infty} = \sum_{0 \leq n} p(n) q^n$$

Theorem 15.17 (Ramanujan の合同式)**.**

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

Theorem 15.18 ($n=\infty$ における漸近表示)**.**

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

15.6 Sum of squares function

Definition 15.19.

$$r_k(n) := \#\{(a_1, \dots, a_k) \in \mathbb{Z}^k; n = a_1^2 + \cdots + a_k^2\}$$

Theorem 15.20.

$$\begin{aligned} r_2(n) &= 4 \sum_{2\nmid d|n} (-1)^{(d-1)/2} \\ r_4(n) &= 8 \sum_{4\nmid d|n} d \\ r_8(n) &= 16 \sum_{d|n} (-1)^{n-d} d^3 \end{aligned}$$

Theorem 15.21.

$$\sum_{0\le n} r_k(n) q^n = \vartheta_3(0,q)^k$$

15.7 Chebyshev 関数

Definition 15.22.

$$\begin{aligned} \vartheta(x) &:= \sum_{p\le x,\, p:\mathrm{prime}} \ln p \\ \psi(x) &:= \sum_{n\le x} \Lambda(n) \end{aligned}$$

Theorem 15.23.

$$\psi(x) = \sum_{0<n} \vartheta\left(x^{1/n}\right)$$

Theorem 15.24.

$$\mathrm{lcm}(1,2,\dots,n) = e^{\psi(n)}$$

Theorem 15.25. ρ は Riemann ゼータ関数の非自明な零点全体を動くとする.

$$\psi(x) = x - \frac{1}{2}\ln\left(1 - \frac{1}{x^2}\right) - \ln 2\pi - \sum_{\rho} \frac{x^\rho}{\rho}$$

15.8 Mertens 関数

Definition 15.26.

$$M(n) := \sum_{k=1}^{n} \mu(k)$$
$$M(x) := M(\lfloor x \rfloor)$$

Theorem 15.27.

$$\frac{1}{\zeta(s)} = s \int_1^\infty \frac{M(x)}{x^{s+1}}\, dx$$

Theorem 15.28. ψ を第 2 種 Chebyshev 関数とする.

$$\psi(x) = \sum_{0<n} M\left(\frac{x}{n}\right) \ln n$$

15.9 素数計数関数

Definition 15.29.

$$\pi(x) := \sum_{p \le x,\, p:\mathrm{prime}} 1$$

Theorem 15.30 (素数定理)**.**

$$\lim_{x\to\infty} \frac{\pi(x) \ln x}{x} = 1$$

15.10 素数ゼータ関数

Definition 15.31.

$$P(s) := \sum_{p:\mathrm{prime}} \frac{1}{p^s}$$

Theorem 15.32.

$$\ln \zeta(s) = \sum_{0<n} \frac{P(ns)}{n}$$
$$P(s) = \sum_{0<n} \frac{\mu(n)}{n} \ln \zeta(ns)$$

15.11 Riesz 関数

Definition 15.33.

$$\mathrm{Riesz}(x) := \sum_{0<n} \frac{(-1)^{n-1}}{(n-1)!\zeta(2n)} x^n$$

Theorem 15.34 (級数表示)**.**

$$\mathrm{Riesz}(x) = \frac{6x}{\pi^2} + x \sum_{0<n} \frac{\mu(n)}{n^2}(e^{-x/n^2} - 1)$$

Theorem 15.35 (Mellin 変換)**.**

$$\int_0^\infty x^{s-1}\,\mathrm{Riesz}(x)\,dx = \frac{\Gamma(s+1)}{\zeta(-2s)}$$

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