

A q -ANALOGUE OF LI'S THEOREM AND THE AOKI-OHNO RELATION

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ABSTRACT. Li's theorem generalizes the Kaneko–Ohno duality for multiple zeta-star values. In this paper, using a result of Takeyama and a summation formula for basic hypergeometric series, we establish a q -analogue of Li's explicit gamma-function evaluation. As a consequence, we obtain an alternative proof of Li's theorem. We also establish a q -analogue of the Aoki–Ohno relation for multiple zeta-star values.

1. INTRODUCTION

Given an index $\mathbf{k} := (k_1, \dots, k_r) \in (\mathbb{Z}_{>0})^r$ with $k_r \geq 2$ and a complex number q satisfying $|q| < 1$, we define the multiple zeta-star values and their q -analogues by

$$\begin{aligned}\zeta^\star(\mathbf{k}) &:= \sum_{0 < n_1 \leq \dots \leq n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}, \\ \zeta_q^\star(\mathbf{k}) &:= \sum_{0 < n_1 \leq \dots \leq n_r} \frac{q^{n_1(k_1-1)}}{(1-q^{n_1})^{k_1}} \dots \frac{q^{n_r(k_r-1)}}{(1-q^{n_r})^{k_r}}.\end{aligned}$$

We also define

$$\begin{aligned}\Phi^\star(u, v, w) &:= \sum_{0 < s \leq r \leq k} \left(\sum_{\mathbf{k} \in I_0(k, r, s)} \zeta^\star(\mathbf{k}) \right) u^{k-r-s} v^{r-s} w^{s-1}, \\ \Phi_q^\star(u, v, w) &:= \sum_{0 < s \leq r \leq k} \left(\sum_{\mathbf{k} \in I_0(k, r, s)} \zeta_q^\star(\mathbf{k}) \right) u^{k-r-s} v^{r-s} w^{s-1}\end{aligned}$$

as formal power series in u, v and w , where

$$I_0(k, r, s) := \{(k_1, \dots, k_r) \in (\mathbb{Z}_{>0})^r; k_1 + \dots + k_r = k, \#\{1 \leq i \leq r; k_i \geq 2\} = s, k_r \geq 2\}.$$

Sums of multiple zeta values with prescribed weight, depth and height were systematically studied through generating functions by Ohno and Zagier [OZ]. In the star case, Aoki and Ohno [AO] obtained an explicit formula for sums with fixed weight and height, and Aoki, Kombu and Ohno [AKO] subsequently studied the three-variable generating function $\Phi^\star(u, v, w)$. Kaneko and Ohno [KO] proved a duality-type relation for multiple zeta-star values of height one and conjectured its extension to arbitrary height.

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Yamazaki [Y] gave an alternative proof of the height-one case using the generating function of Aoki, Kombu and Ohno. The conjecture in full generality was proved by Li [L], who established the stronger gamma-function identity stated below.

Theorem 1.1 (Li [L]). *Let a and b satisfy $a + b = -u + v$ and $ab = -uv - w$. Then the following identity holds:*

$$\begin{aligned} & u\Phi^*(-u, v, w) - v\Phi^*(-v, u, w) \\ &= \frac{u-v}{ab} + A(u, v, a, b) \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)}, \end{aligned}$$

where

$$A(u, v, a, b) := \frac{1}{2\pi} \left(\frac{\cos \pi u}{\sin \pi v} - \frac{\cos \pi v}{\sin \pi u} + \cos \pi(a-b)(\cot \pi u - \cot \pi v) \right).$$

For complex numbers $a, a_1, \dots, a_r$ and a nonnegative integer n , we define the q -shifted factorials by

$$\begin{aligned} (a; q)_n &:= \begin{cases} 1 & n = 0 \\ (1-a)(1-aq) \cdots (1-aq^{n-1}) & n > 0 \end{cases}, \\ (a; q)_{-n} &:= \frac{(-a)^{-n} q^{\binom{n+1}{2}}}{(q/a; q)_n}, \\ (a; q)_\infty &:= \prod_{j=0}^{\infty} (1-aq^j). \end{aligned}$$

For $n \in \mathbb{Z} \cup \{\infty\}$, we also set

$$(a_1, \dots, a_r; q)_n := (a_1; q)_n \cdots (a_r; q)_n.$$

Several q -analogues of multiple zeta values and multiple zeta-star values have been studied, and different q -models are suited to different classical identities; see, for example, [OO, OT, OÖZ]. For the non-strict model considered here, Takeyama [T] expressed the generating function $\Phi_q^*(u, v, w)$ in terms of a basic hypergeometric series. In the present paper, we combine Takeyama's formula with a limiting case of Bailey's ${}_6\psi_6$ summation to obtain a q -analogue of Li's gamma-function identity. After a suitable rescaling of the variables, its limit as $q \rightarrow 1^-$ recovers Li's identity, while a specialization of the same generating function yields a q -analogue of the Aoki–Ohno relation.

Theorem 1.2. *Let α and β satisfy $\alpha + \beta = \frac{1+w}{1-v} + 1 - u$ and $\alpha\beta = \frac{1-u}{1-v}$. Then the following identity holds:*

$$\begin{aligned} u\Phi_q^*(-u, v, w) - v\Phi_q^*(-v, u, w) &= \frac{v-u}{uv+w} - \frac{\left(\frac{1-u}{1-v}, \frac{1-v}{1-u}q, \frac{\alpha q}{1-u}, \frac{\beta q}{1-u}, q, q; q\right)_\infty}{\left(\alpha, q/\alpha, \beta, q/\beta, \frac{q}{1-u}, \frac{q}{1-v}; q\right)_\infty} \\ &= \frac{v-u}{uv+w} (1 - \exp(\Omega_q(u, v, w))) \end{aligned}$$

where $\zeta_q(k) := \zeta_q^*(k)$ for $k \geq 2$, and

$$\begin{aligned} L_q(X) &:= \sum_{2 \leq r} \frac{X^r}{r} \sum_{k=2}^r (-1)^{r-k} \zeta_q(k) \\ \Omega_q(u, v, w) &:= -L_q\left(\frac{v-u}{1-v}\right) - L_q\left(\frac{u-v}{1-u}\right) + L_q(\alpha-1) + L_q(\alpha^{-1}-1) \\ &\quad + L_q(\beta-1) + L_q(\beta^{-1}-1) - L_q\left(\frac{\alpha}{1-u}-1\right) \\ &\quad - L_q\left(\frac{\beta}{1-u}-1\right) + L_q\left(\frac{u}{1-u}\right) + L_q\left(\frac{v}{1-v}\right) \end{aligned}$$

whenever all expressions involved are well-defined. In particular, for any positive integers m, n and s with $m, n \geq s$, we have

$$(-1)^m \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta_q^*(\mathbf{k}) - (-1)^n \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta_q^*(\mathbf{k}) \in \mathbb{Q}[\zeta_q(2), \zeta_q(3), \dots].$$

By setting $w = 0$ in Theorem 1.2, we obtain a q -analogue of the Kaneko–Ohno duality [KO].

Corollary 1.3. *The following identity holds:*

$$u\Phi_q^*(-u, v, 0) - v\Phi_q^*(-v, u, 0) = \frac{v-u}{uv} - \frac{\left(\frac{1-u}{1-v}, \frac{1-v}{1-u}q, \frac{q}{(1-u)(1-v)}, q, q, q; q\right)_\infty}{\left(1-u, \frac{q}{1-u}, \frac{q}{1-u}, (1-v)q, \frac{1}{1-v}, \frac{q}{1-v}; q\right)_\infty}.$$

In particular, for any positive integers m, n , we have

$$(-1)^m \zeta_q^*(\{1\}^n, m+1) - (-1)^n \zeta_q^*(\{1\}^m, n+1) \in \mathbb{Q}[\zeta_q(2), \zeta_q(3), \dots],$$

where $\{1\}^r$ denotes the index consisting of r copies of 1.

Aoki and Ohno [AO] proved the following identity.

Theorem 1.4 (Aoki–Ohno [AO]). *Let k and s be positive integers satisfying $k \geq 2s$. Then the following identity holds:*

$$\sum_{r=1}^k \sum_{\mathbf{k} \in I_0(k, r, s)} \zeta_q^*(\mathbf{k}) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) \zeta(k).$$

This relation can be written as

$$\Phi^*(u, u, w) = 2 \sum_{0 < n} \frac{(-1)^{n-1}}{(n-u)^2 - w}.$$

We establish the following q -analogue of the Aoki–Ohno relation.

Theorem 1.5. *As a formal power series in u and w , the identity*

$$\Phi_q^*\left(u, \frac{u}{1+u}, w\right) = \sum_{0 < n} \frac{(-1)^{n-1} q^{\binom{n}{2}} (1 + (1+u)q^n)}{(1 - (1+u)q^n)^2 - w(1+u)q^n} ((1+u)q)^n$$

holds.

Indeed, making the substitutions $u \mapsto q^{-u} - 1$ and $w \mapsto (1 - q^w)^2$, multiplying both sides by $(1 - q)^2$, and then taking the coefficientwise limit as $q \rightarrow 1^-$, we recover the generating-function form of Theorem 1.4 with w replaced by w^2 .

This paper is organized as follows. In Section 2, we prove Theorems 1.2 and 1.5 using a result of Takeyama and a summation formula for basic hypergeometric series. In Section 3, we derive Theorem 1.1 from Theorem 1.2 by taking the limit $q \rightarrow 1^-$.

2. PROOFS OF THEOREMS 1.2 AND 1.5

The following formula is a q -analogue of the Aoki–Kombu–Ohno formula [AKO]. It was proved by Takeyama [T].

Lemma 2.1 (Takeyama [T, Theorem 1.1]). *As formal power series in u, v and w , the identity*

$$\Phi_q^*(u, v, w) = \sum_{0 \leq n} \frac{(q, (1+u)q; q)_{n-1} q^n}{\prod_{k=1}^n ((1-v-q^k)(1-(1+u)q^k) - wq^k)}$$

holds.

For later use, we rewrite Takeyama's formula in a form suitable for applying summation formulas for basic hypergeometric series.

Lemma 2.2. *As formal power series in u, v and w , the identity*

$$\Phi_q^*(u, v, w) = \sum_{0 \leq n} \frac{(-1)^{n-1} q^{\binom{n}{2}} (1-v-(1+u)q^{2n})}{(1-v-q^n)(1-(1+u)q^n) - wq^n} \frac{((1+u)q; q)_{n-1}}{\left(\frac{q}{1-v}; q\right)_n} \left(\frac{q}{1-v}\right)^n$$

holds.

Proof. Using a partial fraction decomposition with respect to w , we obtain

$$\begin{aligned} & \sum_{0 \leq n} \frac{(q, (1+u)q; q)_{n-1} q^n}{\prod_{k=1}^n ((1-v-q^k)(1-(1+u)q^k) - wq^k)} \\ &= \sum_{0 \leq n} \frac{1}{(1-v-q^n)(1-(1+u)q^n)q^{-n} - w} \\ & \quad \cdot \sum_{n \leq k} \frac{(q, (1+u)q; q)_{k-1} q^k}{q^{\binom{k+1}{2}} \prod_{\substack{1 \leq j \leq k \\ j \neq n}} ((1-v-q^j)(1-(1+u)q^j)q^{-j} - (1-v-q^n)(1-(1+u)q^n)q^{-n})} \\ &= \sum_{0 \leq n} \frac{1}{(1-v-q^n)(1-(1+u)q^n) - wq^n} \sum_{n \leq k} \frac{(q, (1+u)q; q)_{k-1} q^k}{\prod_{\substack{1 \leq j \leq k \\ j \neq n}} (1-q^{j-n})(1-v-(1+u)q^{j+n})}. \end{aligned}$$

Applying Heine's summation formula (see Gasper and Rahman [GR, §1.5]), we obtain

$$\begin{aligned}
& \sum_{n \leq k} \frac{(q, (1+u)q; q)_{k-1} q^k}{\prod_{\substack{1 \leq j \leq k \\ j \neq n}} (1 - q^{j-n}) (1 - v - (1+u)q^{j+n})} \\
&= \frac{(q, (1+u)q; q)_{n-1}}{(q^{1-n}, \frac{1+u}{1-v} q^{n+1}; q)_{n-1}} \frac{q^n}{(1-v)^{n-1}} \sum_{0 \leq k} \frac{(q^n, (1+u)q^n; q)_k}{(q, \frac{1+u}{1-v} q^{2n+1}; q)_k} \left(\frac{q}{1-v} \right)^k \\
&= \frac{(q, (1+u)q; q)_{n-1}}{(q^{1-n}, \frac{1+u}{1-v} q^{n+1}; q)_{n-1}} \frac{q^n}{(1-v)^{n-1}} \frac{(\frac{1+u}{1-v} q^{n+1}; q)_n}{(\frac{q}{1-v}; q)_n} \\
&= \frac{(-1)^{n-1} q^{\binom{n}{2}} (1 - v - (1+u)q^{2n}) ((1+u)q; q)_{n-1}}{(\frac{q}{1-v}; q)_n} \left(\frac{q}{1-v} \right)^n.
\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.5. Substituting $v = \frac{u}{1+u}$ into the identity in Lemma 2.2, we obtain

$$\begin{aligned}
\Phi_q^* \left(u, \frac{u}{1+u}, w \right) &= \sum_{0 \leq n} \frac{(-1)^{n-1} q^{\binom{n}{2}} (\frac{1}{1+u} - (1+u)q^{2n})}{(\frac{1}{1+u} - q^n) (1 - (1+u)q^n) - wq^n} \frac{((1+u)q; q)_{n-1}}{((1+u)q; q)_n} ((1+u)q)^n \\
&= \sum_{0 \leq n} \frac{(-1)^{n-1} q^{\binom{n}{2}} (1 + (1+u)q^n)}{(1 - (1+u)q^n)^2 - w(1+u)q^n} ((1+u)q)^n.
\end{aligned}$$

$\square$

To prove Theorem 1.2, we use the following limiting case of Bailey's summation formula.

Lemma 2.3. *The identity*

$$\sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\binom{n}{2}} (1 - aq^{2n}) (b, c, d; q)_n}{(1 - a)(aq/b, aq/c, aq/d; q)_n} \left(\frac{a^2 q}{bcd} \right)^n = \frac{(aq, aq/bc, aq/bd, aq/cd, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, q/b, q/c, q/d; q)_\infty}$$

holds.

Proof. Bailey's summation formula [GR, §5.3] states that

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}} \frac{(1 - aq^{2n}) (b, c, d, e; q)_n}{(1 - a)(aq/b, aq/c, aq/d, aq/e; q)_n} \left(\frac{a^2 q}{bcde} \right)^n \\
&= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, a^2 q/bcde; q)_\infty}.
\end{aligned}$$

Letting $e \rightarrow \infty$ in this identity yields the desired result. $\square$

Proof of Theorem 1.2. Applying Lemma 2.2, we have

$$\begin{aligned}
& u\Phi_q^*(-u, v, w) - v\Phi_q^*(-v, u, w) \\
&= \sum_{0 < n} \frac{(-1)^{n-1} q^{\binom{n}{2}} (1-v-(1-u)q^{2n})}{(1-v-q^n)(1-(1-u)q^n) - wq^n} \frac{(1-u; q)_n}{\left(\frac{q}{1-v}; q\right)_n} \left(\frac{q}{1-v}\right)^n \\
&\quad - \sum_{0 < n} \frac{(-1)^{n-1} q^{\binom{n}{2}} (1-u-(1-v)q^{2n})}{(1-u-q^n)(1-(1-v)q^n) - wq^n} \frac{(1-v; q)_n}{\left(\frac{q}{1-u}; q\right)_n} \left(\frac{q}{1-u}\right)^n \\
&= \sum_{0 < n} \frac{(-1)^{n-1} q^{\binom{n}{2}} (1-v-(1-u)q^{2n})}{(1-v-q^n)(1-(1-u)q^n) - wq^n} \frac{(1-u; q)_n}{\left(\frac{q}{1-v}; q\right)_n} \left(\frac{q}{1-v}\right)^n \\
&\quad + \sum_{n < 0} \frac{(-1)^{n-1} q^{\binom{n}{2}} (1-v-(1-u)q^{2n})}{(1-v-q^n)(1-(1-u)q^n) - wq^n} \frac{(1-u; q)_n}{\left(\frac{q}{1-v}; q\right)_n} \left(\frac{q}{1-v}\right)^n \\
&= \frac{v-u}{uv+w} - \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\binom{n}{2}} (1-v-(1-u)q^{2n})}{(1-v-q^n)(1-(1-u)q^n) - wq^n} \frac{(1-u; q)_n}{\left(\frac{q}{1-v}; q\right)_n} \left(\frac{q}{1-v}\right)^n.
\end{aligned}$$

Applying Lemma 2.3, we obtain

$$\begin{aligned}
& \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\binom{n}{2}} (1-v-(1-u)q^{2n})}{(1-v-q^n)(1-(1-u)q^n) - wq^n} \frac{(1-u; q)_n}{\left(\frac{q}{1-v}; q\right)_n} \left(\frac{q}{1-v}\right)^n \\
&= \frac{1}{(1-\alpha)(1-\beta)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\binom{n}{2}} (1-\frac{1-u}{1-v}q^{2n})}{(\alpha q, \beta q, \frac{q}{1-v}; q)_n} \frac{(\alpha, \beta, 1-u; q)_n}{\left(\frac{q}{1-v}\right)^n} \left(\frac{q}{1-v}\right)^n \\
&= \frac{\left(\frac{1-u}{1-v}, \frac{1-v}{1-u}q, \frac{\alpha q}{1-u}, \frac{\beta q}{1-u}, q, q; q\right)_\infty}{\left(\alpha, q/\alpha, \beta, q/\beta, \frac{q}{1-u}, \frac{q}{1-v}; q\right)_\infty}.
\end{aligned}$$

Next, we prove the second equality. As an identity of formal power series in X , we have

$$\begin{aligned}
\frac{((1-X)q; q)_\infty}{(q; q)_\infty} &= \prod_{0 < n} \left(1 + \frac{Xq^n}{1-q^n}\right) \\
&= \exp \left(\sum_{0 < n} \ln \left(1 + \frac{Xq^n}{1-q^n}\right) \right) \\
&= \exp \left(\sum_{0 < n, r} \frac{(-1)^{r-1} X^r}{r} \left(\frac{q^n}{1-q^n}\right)^r \right).
\end{aligned}$$

Using the following identity, which is easily proved by induction on r

$$\sum_{0 < n} \left(\frac{q^n}{1-q^n}\right)^r = \sum_{k=2}^r (-1)^{r-k} \zeta_q(k) + (-1)^{r-1} \sum_{0 < n} \frac{q^n}{1-q^n},$$

we obtain

$$\begin{aligned} \frac{((1-X)q; q)_\infty}{(q; q)_\infty} &= \exp \left(\sum_{0 < r} \frac{(-1)^{r-1} X^r}{r} \left(\sum_{k=2}^r (-1)^{r-k} \zeta_q(k) + (-1)^{r-1} \sum_{0 < n} \frac{q^n}{1-q^n} \right) \right) \\ &= \exp \left(-L_q(-X) - \ln(1-X) \sum_{0 < n} \frac{q^n}{1-q^n} \right). \end{aligned}$$

Applying this identity, we obtain

$$\begin{aligned} &\frac{\left(\frac{1-u}{1-v}, \frac{1-v}{1-u} q, \frac{\alpha q}{1-u}, \frac{\beta q}{1-u}, q, q; q \right)_\infty}{\left(\alpha, q/\alpha, \beta, q/\beta, \frac{q}{1-u}, \frac{q}{1-v}; q \right)_\infty} \\ &= \frac{1 - \frac{1-u}{1-v}}{(1-\alpha)(1-\beta)} \frac{\left(\frac{1-u}{1-v} q, \frac{1-v}{1-u} q, \frac{\alpha q}{1-u}, \frac{\beta q}{1-u}, q, q; q \right)_\infty}{\left(\alpha q, q/\alpha, \beta q, q/\beta, \frac{q}{1-u}, \frac{q}{1-v}; q \right)_\infty} \\ &= \frac{v-u}{uv+w} \exp(\Omega_q(u, v, w)). \end{aligned}$$

Set $R_q := \mathbb{Q}[\zeta_q(2), \zeta_q(3), \dots]$. Since Ω_q is symmetric in α and β , and since

$$\alpha + \beta, \alpha\beta \in R_q[[u, v, w]],$$

we have $\Omega_q(u, v, w) \in R_q[[u, v, w]]$. When $w = -uv$, we may take $\alpha = 1, \beta = \frac{1-u}{1-v}$. Direct substitution then gives $\Omega_q(u, v, -uv) = 0$. Hence, $uv + w$ divides $\Omega_q(u, v, w)$ in $R_q[[u, v, w]]$. Since $1 - \exp(\Omega_q)$ is divisible by Ω_q , it follows that

$$\frac{v-u}{uv+w} (1 - \exp(\Omega_q(u, v, w))) \in R_q[[u, v, w]].$$

The coefficient of $u^{m-s+1}v^{n-s+1}w^{s-1}$ on the left-hand side is

$$(-1)^{m-s} \sum_{\mathbf{k} \in I_0(m+n+1, n+1, s)} \zeta_q^*(\mathbf{k}) - (-1)^{n-s} \sum_{\mathbf{k} \in I_0(m+n+1, m+1, s)} \zeta_q^*(\mathbf{k}).$$

Multiplying by $(-1)^s$ gives the last assertion. $\square$

3. DERIVATION OF THEOREM 1.1 FROM THEOREM 1.2

In this section, we derive Theorem 1.1 as a limiting case of Theorem 1.2. Throughout this section, we assume that $0 < q < 1$. For $\mathbf{k} \in I_0(k, r, s)$, we have

$$\begin{aligned} &\lim_{q \rightarrow 1^-} (1-q)^2 (1-q^u)^{k-r-s} (1-q^v)^{r-s} (1-q^w)^{2s-2} \zeta_q^*(\mathbf{k}) \\ &= \lim_{q \rightarrow 1^-} \left(\frac{1-q^u}{1-q} \right)^{k-r-s} \left(\frac{1-q^v}{1-q} \right)^{r-s} \left(\frac{1-q^w}{1-q} \right)^{2s-2} (1-q)^k \zeta_q^*(\mathbf{k}) \\ &= u^{k-r-s} v^{r-s} w^{2s-2} \zeta^*(\mathbf{k}). \end{aligned}$$

Hence, we obtain

$$\lim_{q \rightarrow 1^-} (1-q)^2 \Phi_q^*(1-q^u, 1-q^v, (1-q^w)^2) = \Phi^*(u, v, w^2).$$

Put $U_q = 1 - q^u$, $V_q = 1 - q^v$, $W_q = 1 - q^w$. Then, by the same coefficientwise argument,

$$\begin{aligned} & \lim_{q \rightarrow 1^-} (1 - q)(U_q \Phi_q^*(-U_q, V_q, W_q^2) - V_q \Phi_q^*(-V_q, U_q, W_q^2)) \\ &= u \Phi^*(-u, v, w^2) - v \Phi^*(-v, u, w^2). \end{aligned}$$

Moreover,

$$\lim_{q \rightarrow 1^-} (1 - q) \frac{V_q - U_q}{U_q V_q + W_q^2} = \frac{v - u}{uv + w^2} = \frac{u - v}{ab}.$$

Thus, it remains to prove the following proposition.

Proposition 3.1. *Let $\alpha = \alpha(q)$ and $\beta = \beta(q)$ satisfy*

$$\alpha + \beta = q^{-v}(1 + (1 - q^w)^2) + q^u, \quad \alpha\beta = q^{u-v},$$

and let a and b satisfy

$$a + b = -u + v, \quad ab = -uv - w^2.$$

Then the following identity holds:

$$\begin{aligned} & \lim_{q \rightarrow 1^-} (1 - q) \frac{(q^{u-v}, q^{1-u+v}, \alpha q^{1-u}, \beta q^{1-u}, q, q; q)_\infty}{(\alpha, q/\alpha, \beta, q/\beta, q^{1-u}, q^{1-v}; q)_\infty} \\ &= -A(u, v, a, b) \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)}. \end{aligned}$$

Proof. Write $\alpha = q^{-a'}$, $\beta = q^{-b'}$. Using the q -gamma function (see Gasper and Rahman [GR, §1.10]), we rewrite the left-hand side of the identity as

$$\begin{aligned} & (1 - q) \frac{(q^{u-v}, q^{1-u+v}, \alpha q^{1-u}, \beta q^{1-u}, q, q; q)_\infty}{(\alpha, q/\alpha, \beta, q/\beta, q^{1-u}, q^{1-v}; q)_\infty} \\ &= \frac{\Gamma_q(-a')\Gamma_q(a'+1)\Gamma_q(-b')\Gamma_q(b'+1)\Gamma_q(1-u)\Gamma_q(1-v)}{\Gamma_q(u-v)\Gamma_q(1-u+v)\Gamma_q(1-u-a')\Gamma_q(1-u-b')}. \end{aligned}$$

Under the conditions

$$q^{-a'} + q^{-b'} = q^{-v}(1 + (1 - q^w)^2) + q^u, \quad q^{-a'-b'} = q^{u-v},$$

we choose continuous functions $a' = a'(q)$ and $b' = b'(q)$, defined for q sufficiently close to 1, such that $a' + b' = -u + v = a + b$. It is straightforward to verify that the limits $\lim_{q \rightarrow 1^-} a'$ and $\lim_{q \rightarrow 1^-} b'$ exist. We then have

$$\begin{aligned} \lim_{q \rightarrow 1^-} (v - a')(v - b') &= \lim_{q \rightarrow 1^-} \frac{1 - q^{v-a'}}{1 - q} \frac{1 - q^{v-b'}}{1 - q} \\ &= \lim_{q \rightarrow 1^-} \frac{q^v(q^{-v} - q^{-a'} - q^{-b'} + q^u)}{(1 - q)^2} \\ &= - \lim_{q \rightarrow 1^-} \frac{(1 - q^w)^2}{(1 - q)^2} \\ &= -w^2. \end{aligned}$$

Thus, $\lim_{q \rightarrow 1^-} a'b' = -uv - w^2 = ab$. After interchanging a' and b' if necessary, we may therefore assume that $\lim_{q \rightarrow 1^-} a' = a$ and $\lim_{q \rightarrow 1^-} b' = b$. It follows that

$$\begin{aligned}
& \lim_{q \rightarrow 1^-} (1-q) \frac{(q^{u-v}, q^{1-u+v}, \alpha q^{1-u}, \beta q^{1-u}, q, q; q)_\infty}{(\alpha, q/\alpha, \beta, q/\beta, q^{1-u}, q^{1-v}; q)_\infty} \\
&= \lim_{q \rightarrow 1^-} \frac{\Gamma_q(-a')\Gamma_q(a'+1)\Gamma_q(-b')\Gamma_q(b'+1)\Gamma_q(1-u)\Gamma_q(1-v)}{\Gamma_q(u-v)\Gamma_q(1-u+v)\Gamma_q(1-u-a')\Gamma_q(1-u-b')} \\
&= \frac{\Gamma(-a)\Gamma(a+1)\Gamma(-b)\Gamma(b+1)\Gamma(1-u)\Gamma(1-v)}{\Gamma(u-v)\Gamma(1-u+v)\Gamma(1-u-a)\Gamma(1-u-b)} \\
&= \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)} \frac{\sin \pi(u-v) \sin \pi(u+a) \sin \pi(u+b)}{\pi \sin \pi u \sin \pi v}.
\end{aligned}$$

A direct calculation gives

$$\begin{aligned}
& \frac{\sin \pi(u-v) \sin \pi(u+a) \sin \pi(u+b)}{\pi \sin \pi u \sin \pi v} \\
&= \frac{\sin \pi(u-v)(\cos \pi(a-b) - \cos \pi(u+v))}{2\pi \sin \pi u \sin \pi v} \\
&= \frac{1}{2\pi} \left(\cos \pi(a-b)(\cot \pi v - \cot \pi u) - \frac{\sin \pi(u-v) \cos \pi(u+v)}{\sin \pi u \sin \pi v} \right) \\
&= \frac{1}{2\pi} \left(\cos \pi(a-b)(\cot \pi v - \cot \pi u) - \frac{\sin 2\pi u - \sin 2\pi v}{2 \sin \pi u \sin \pi v} \right) \\
&= \frac{1}{2\pi} \left(\cos \pi(a-b)(\cot \pi v - \cot \pi u) - \frac{\cos \pi u}{\sin \pi v} + \frac{\cos \pi v}{\sin \pi u} \right) \\
&= -A(u, v, a, b).
\end{aligned}$$

This completes the proof. $\square$

Combining Theorem 1.2 with Proposition 3.1, we obtain Theorem 1.1 with w replaced by w^2 . Replacing w^2 by w completes the proof.

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